2.2 Quadratic functions

Textbook Sections 2.3, 2.4 and 2.6

2.2.1 Case study: *How to select the optimal pricing of Donelly's chocolates?*

Following on from our previous case study, we now look at how Donelly's can determine what the optimal price for their chocolates should be. The owners decide to conduct a market analysis to determine how many chocolates their customers would buy as a function of their price. After interviewing all of their customers over the course of a week, they estimate that the number of chocolates n they would be able to sell per day as a function of their price p would be $n(p) = 1000 - 200p$. Based on this information, and using the cost analysis from the previous study, what would be optimal price for the chocolates? What is the predicted net profit at that price?

We saw in last lecture that $\begin{cases} \text{money in} = pn \\ \text{money out} = 500 + n \end{cases}$

$\rightarrow$ Net profit is money in − money out = $pn - (500+n)$

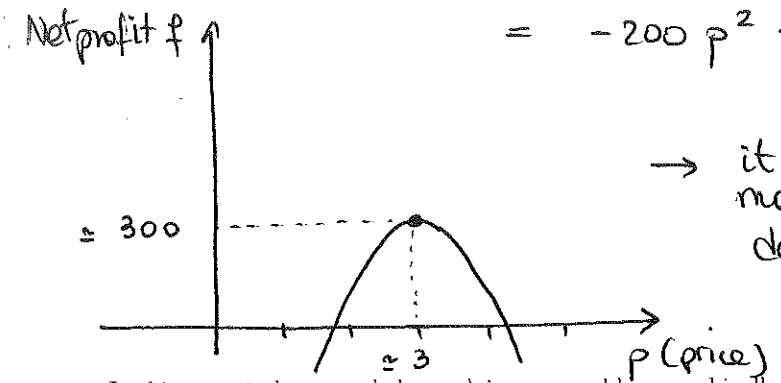
In this lecture we learn n is a function of price p : so

$$\text{Net profit} = p\,n(p) - 500 - n(p) = (p-1)n(p) - 500$$

is a function of price: $f(p) = (p-1)(1000-200p) - 500$

$$= 1000p - 200p^2 - 1000 + 200p - 500$$

$$= -200p^2 + 1200p - 1500$$



$\rightarrow$ it looks like the maximum Net profit per day is \$300/day, when the chocolates are priced at \$3/chocolate.

In this case study, we ended up solving our problem graphically. However, with the right mathematical tools, we can also do it without graphs. Let's learn more about quadratics.

2.2.2 Mathematical corner: Properties of quadratic functions

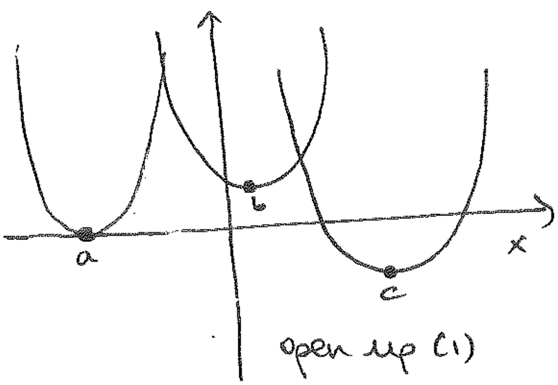
A quadratic function is a special type of polynomial function. The general expression for a quadratic function is

$$f(x) = ax^2 + bx + c \qquad a, b, c \text{ real numbers.}$$

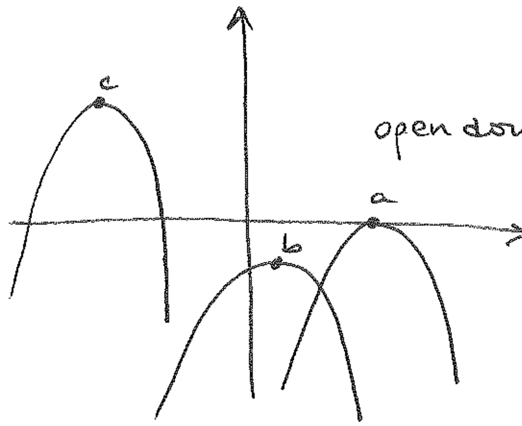
The domain of a quadratic function is $\mathcal{D} = \mathbb{R}$.

The graph of all quadratic functions is called a *parabola.* The exact shape and position of the parabola depends on the coefficients of the quadratic. Different cases can arise:

- The parabola can open up (1) or down (2)
- The parabola can cross, touch or not cross the x-axis
  (c) (a) (b)



open up (1)



open down (2)

Opening up or down. Whether a parabola opens "up" or "down" can very easily be determined simply by inspection of the quadratic term ax^2 in the function.

Let's consider two examples of quadratic functions:

- $f(x) = 3x^2 - 2x - 1$
- $g(x) = -2x^2 + x + 1$

and graph them on Wolfram Alpha. We also compare their graphs with those of the functions $3x^2$ and $-2x^2$. We notice that:

- The graph of $f(x) = 3x^2 - 2x - 1$ looks like the graph of $y = 3x^2$ for large x (negative or positive)
- The graph of $g(x) = -2x^2 + x + 1$ similarly looks like the graph of $y = -2x^2$ for large x

This is in fact true of all quadratics!

- The graph of $f(x) = ax^2 + bx + c$ looks like the graph of $y = ax^2$ for large x
- This implies that the graph of $ax^2 + bx + c$ opens up if $a > 0$, and down if $a < 0$

$\cup$ positive a $\qquad$ $\cap$ negative a.

BEHAVIOR NEAR THE y-AXIS What the parabola looks like near the $y-$ axis (i.e. when x is close to 0) can *also* very easily be determined simply by inspection of the quadratic function, but this time, of the $bx+c$ bit.

Let's consider the functions $f(x)$ and $g(x)$ again, but this time zoom in the graph near $x = 0$. We also compare their graphs with those of the functions $-2x-1$ and $x+1$. We notice that:

- The graph of $f(x) = 3x^2 - 2x - 1$ looks like the line $y = -2x - 1$ for small x (x near 0)
- The graph of $f(x) = -2x^2 + x + 1$ looks like the line $y = x + 1$ for small x

Again, this is true for every quadratic!

- The graph of $f(x) = ax^2 + bx + c$ looks like the line $y = bx + c$ for small x
- The line $y = bx + c$ is <u>tangent</u> to the parabola at $x = 0$.

VERTEX OF A PARABOLA AND VERTEX FORM

- The vertex of a parabola is the position (x & y-coordinate) of its extremum (minimum or maximum). It is given by $x_v = -\frac{b}{2a}$ $y_v = f(x_v) = ax_v^2 + bx_v + c$
- The equation of the quadratic can be put into <u>vertex form</u>: $f(x) = ax^2 + bx + c = a(x - x_v)^2 + y_v$

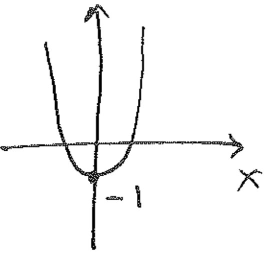
We already saw in the previous case study an example where we were interested in finding out what the coordinates of the vertex are (in that case, where the maximum of the graph was). We can now check analytically our graphical result. Indeed, $f(p) = -200p^2 + 1200p - 1500$

The position of the vertex is given by

$$p_v = -\frac{b}{2a} = \frac{-1200}{2(-200)} = \frac{-1200}{-400} = \boxed{3} \rightarrow \text{price is } \$3$$

$$f(p_v) = -200(p_v^2) + 1200p_v - 1500$$
$$= -200(9) + 1200(3) - 1500 = \boxed{300} \rightarrow \text{profit is } \$300$$

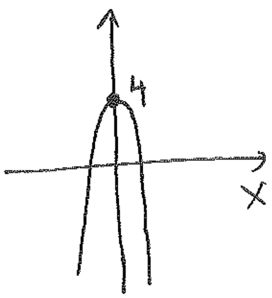
Here are further examples. In each case, let's use the vertex coordinates formula to find the position of the vertex, then use the vertex form of the quadratic to sketch the graph of the function using basic transformations.



- $f(x) = x^2 - 1$ $x_v = -\frac{b}{2a} = 0$ $y_v = 0^2 - 1 = -1$

Vertex form: $a(x-x_v)^2 + y_v = 1(x-0)^2 + (-1) = x^2 - 1$

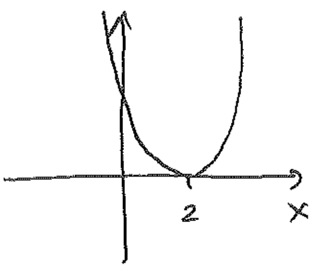
$\rightarrow$ this was already in vertex form



- $f(x) = -3x^2 + 4$ $x_v = -\frac{b}{2a} = -\frac{0}{-3} = 0$ $y_v = -3(0)^2 + 4 = 4$

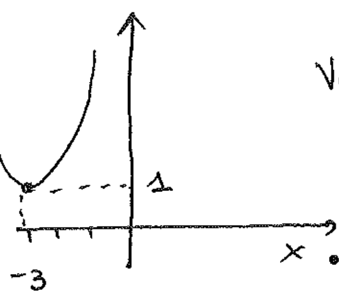
Vertex form: $a(x-x_v)^2 + y_v = -3(x-0)^2 + 4 = -3x^2 + 4$

$\rightarrow$ this was already in vertex form.



- $f(x) = x^2 - 4x + 4$ $x_v = -\frac{b}{2a} = \frac{-(-4)}{2(1)} = \frac{4}{2} = 2$ $y_v = 2^2 - 4(2) + 4 = 0$

Vertex form: $a(x-2)^2 + 0 = (x-2)^2$

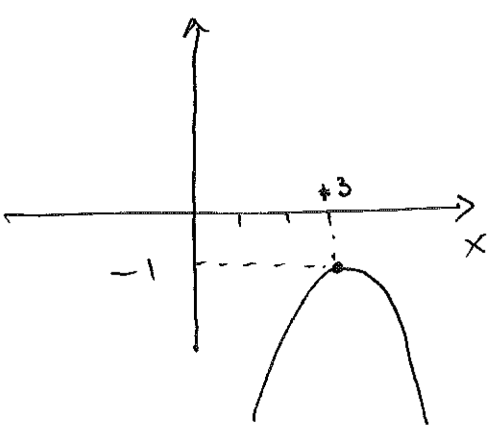


- $f(x) = x^2 + 6x + 10$ $x_v = -\frac{b}{2a} = -\frac{6}{2(1)} = -3$ $y_v = (-3)^2 + 6(-3) + 10 = 1$

Vertex form: $a(x-(-3))^2 + 1 = (x+3)^2 + 1$

- $f(x) = -5x^2 + 30x - 46$ $x_v = \frac{-30}{2(-5)} = 3$ $y_v = -5(3)^2 + 30(3) - 46 = -1$

Vertex form: $a(x-3)^2 + (-1) = -5(x-3)^2 - 1$



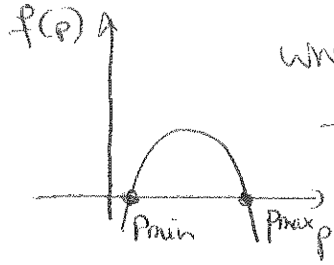
2.2.3 Case study: *How to select the optimal pricing of Donnelly's chocolates?*

In the previous lecture, we learned how varying the selling price of Donnelly's chocolates affects the expected number of chocolates sold in any given day, and therefore the net profit that the company can make. We then selected the pricing to maximize profit. Alternatively, the company may decide to sell the chocolates a little below this price to attract new clients. How low a price can they sell their chocolates without losing any money? Conversely, how high a price could they try to sell them without losing money?

We saw in the last lecture that the net profit is a function of price $f(p) = -200p^2 + 1200p - 1500$

$\rightarrow$ To answer the questions we must find when $f(p) \geq 0$

From the graph we saw that $\boxed{f(p) \geq 0 \text{ if } p_{min} < p < p_{max}}$



where p_{min} & p_{max} are such that $f(p_{min}) = f(p_{max}) = 0$

$\rightarrow$ the x-intercepts of the function f. To find them, use the vertex form: $p_v = 3$, $f(p_v) = 300$ so

$$f(p) = -200(p - p_v)^2 + f(p_v) = -200(p-3)^2 + 300$$

$\rightarrow$ Solve $f(p) = 0 \Rightarrow -200(p-3)^2 + 300 = 0$

$$\Rightarrow (p-3)^2 = -\frac{300}{-200} = \frac{3}{2}$$

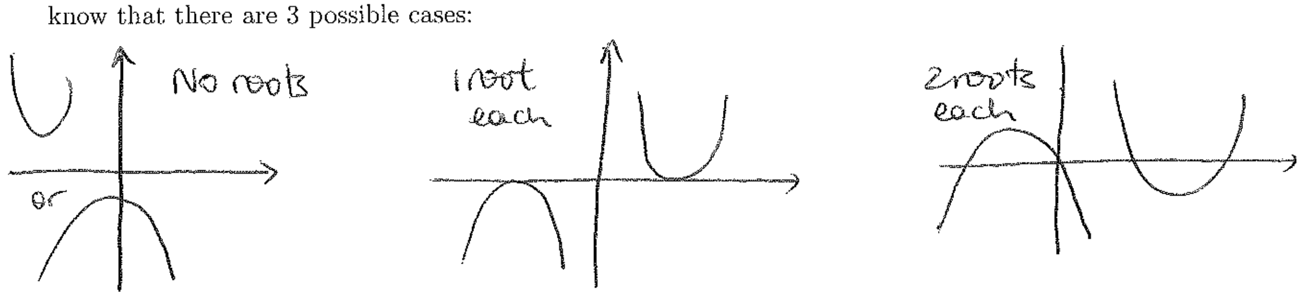
$$\Rightarrow p - 3 = \pm\frac{\sqrt{3}}{\sqrt{2}} \Rightarrow p = 3 \pm \frac{\sqrt{3}}{\sqrt{2}}$$

$\rightarrow p_{min} = 3 - \frac{\sqrt{3}}{\sqrt{2}} \cong 1.775 \qquad p_{max} = 3 + \frac{\sqrt{3}}{\sqrt{2}} \cong 4.225$

2.2.4 Mathematical corner: Roots of quadratics

DEFINITION: The roots of a quadratic functions, also called the x-intercepts of a quadratic function, are the values of x for which $ax^2 + bx + c = 0$.

As we saw in the last lecture, some parabolas cross the $x-$axis, and some do not. Graphically, we know that there are 3 possible cases:



Since a parabola is the graph of a quadratic function, this means that some quadratic functions $f(x) = ax^2 + bx + c$ have 2 roots (i.e. 2 x-intercepts), some have one root (i.e. 1 x-intercepts), and some do not have any. In other words,

- Sometimes there are 2 solutions to the equation $ax^2+bx+c=0$
- " " is 1 " " " "
- " there is No solution " " "

FACTORED FORM: We have the following equivalences:

- $f(x) = ax^2+bx+c$ has a single root $x_1 \Leftrightarrow ax^2+bx+c = a(x-x_1)^2$
- $f(x) = ax^2+bx+c$ has 2 roots x_1 & $x_2 \Leftrightarrow ax^2+bx+c = a(x-x_1)(x-x_2)$

These forms, when they exist, are called "factored form" of the function $f(x)$.

The symbol $\Leftrightarrow$ implies that there is a strict equivalence relationship between the two statements "$f(x)$ can be factored" and "$f(x)$ has roots": the first implies the second, and conversely, the second implies the first. In fact, it is quite easy to check that, in both cases, the second statement implies the first. Indeed,

- If $f(x) = a(x-x_1)^2$ then the solution to $f(x)=0$ is such that $x-x_1=0 \Rightarrow x=x_1 \rightarrow$ 1 root, $x=x_2$
- If $f(x) = a(x-x_1)(x-x_2)$ then the solutions to $f(x)=0$ are $x-x_1=0$ or $x-x_2=0$
  $\Leftrightarrow x=x_1$ or $x=x_2 \rightarrow$ 2 roots, $x=x_1$ or x_2.

Whether a quadratic has one or two roots, and what the roots actually are, is therefore obvious from its factored form.

EXAMPLES

- $f(x) = 3(x-1)(x+2)$. $f(x)=0$ implies $x-1=0$ or $x+2=0$
  $\rightarrow x=1$ or $x=-2$
  $f(x)$ has 2 roots, $x=1$ and $x=-2$
- $f(x) = \frac{1}{2}(x-4)^2$ $f(x)=0$ implies $x-4=0$
  $\rightarrow x=4$.
  $f(x)$ has 1 root, $x=4$.

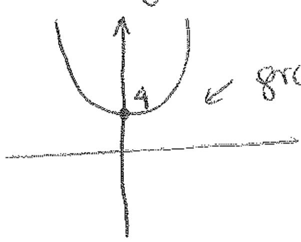
The interesting thing about equivalence statements in logic is that if you have one, then you also have equivalence of the opposites:

f has roots $\Leftrightarrow$ f can be factored also means f has no roots $\Leftrightarrow$ f cannot be factored

EXAMPLES

- $f(x) = x^2 + 4$ $\rightarrow$ sum of two squares, cannot be factored

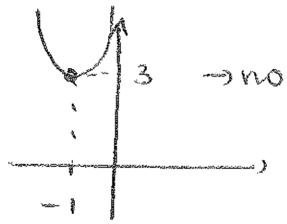
$\Updownarrow$



$\swarrow$ graph has no x-intercepts, f has no roots

- $f(x) = x^2 + 2x + 4$ $= x^2 + 2x + 1 + 3 = (x+1)^2 + 3$ $\rightarrow$ sum of two squares

but also



$\rightarrow$ no x-intercepts, no roots.

$\rightarrow$ cannot be factored

FACTORING QUADRATICS. Based on what we just saw, it would be nice to have simple tricks to

- tell us when a quadratic has roots or not,
- or equivalently, factor the quadratic if it can be factored.

As it turns out, there are a few types of quadratics that can very easily be factored:

- $x^2 + 2ax + a^2 = (x+a)^2$
- $x^2 - 2ax + a^2 = (x-a)^2$
- $x^2 - a^2 = (x+a)(x-a)$

EXAMPLES

- $f(x) = x^2 - 2$ $= x^2 - (\sqrt{2})^2$
  $= (x - \sqrt{2})(x + \sqrt{2})$

- $f(x) = 2x^2 - 3$ $= (\sqrt{2}x)^2 - (\sqrt{3})^2 = (\sqrt{2}x - \sqrt{3})(\sqrt{2}x + \sqrt{3})$

- $f(x) = x^2 + 6x + 9$ $= (x+3)^2$

  (use formula 1 with $a = 3$)

- $f(x) = 2x^2 + 4\sqrt{5}x + 10$ $= 2\left[x^2 + 2\sqrt{5}x + 5\right]$

  $= 2(x+\sqrt{5})^2$

  (use formula 1 with $a = \sqrt{5}$)

- $f(x) = -x^2 + 10x - 25$ $= -(x^2 - 10x + 25)$

  $= -(x-5)^2$

  (use formula 2 with $a = 5$)

On the other hand, not every quadratic is in one of these three "ideal forms". What can we do if it isn't? As it turns out, another nice trick exists in that case, and is called "The quadratic formula".

THE QUADRATIC FORMULA. Given the quadratic $ax^2 + bx + c$,

- Calculate the discriminant $D = b^2 - 4ac$
- If $D < 0$ there are no solutions to the equation $ax^2 + bx + c = 0$, and the quadratic cannot be factored.
- If $D = 0$ there is one solution to the equation $ax^2 + bx + c = 0$, $x_1 = -\frac{b}{2a}$ and the quadratic can be factored as

  $f(x) = a\left(x + \frac{b}{2a}\right)^2 = a(x - x_1)^2$
- If $D > 0$ there are two solutions to the equation $ax^2 + bx + c = 0$, which are $x_{1,2} = \dfrac{-b \pm \sqrt{D}}{2a}$

  and the quadratic can be factored as

$$f(x) = a(x - x_1)(x - x_2) = a\left(x - \frac{-b+\sqrt{D}}{2a}\right)\left(x - \frac{-b-\sqrt{D}}{2a}\right)$$

NOTE: The vertex of a parabola is always half-way between the roots! Indeed,

Half way between roots is $\dfrac{x_1 + x_2}{2} = \left[\dfrac{-b+\sqrt{D}}{2a} + \dfrac{-b-\sqrt{D}}{2a}\right]\dfrac{1}{2}$

$= -\dfrac{b}{2a} = x_v.$

EXAMPLES:

- What are the solutions (if any) to the equation $f(x) = 2x^2 - 3x + 1 = 0$? What is the factored form of f?

$$D = +b^2 - 4ac = (-3)^2 - 4(2)(1) = 9 - 8 = 1$$

$$x_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = \frac{-(-3) \pm \sqrt{1}}{2(2)} = \frac{3 \pm 1}{4} = \begin{cases} 1 \\ \frac{1}{2} \end{cases}$$

So $f(x) = a(x - x_1)(x - x_2) = 2(x-1)(x - \frac{1}{2})$

- What are the solutions (if any) to the equation $f(x) = x^2 + x - 6 = 0$? What is the factored form of f?

$$D = b^2 - 4ac = (1)^2 - 4(1)(-6) = 1 + 24 = 25$$

$$x_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = \frac{-1 \pm \sqrt{25}}{2(1)} = \frac{-1 \pm 5}{2} = \begin{cases} -3 \\ 2 \end{cases}$$

So $f(x) = a(x - x_1)(x - x_2) = (x+3)(x-2)$

- What are the solutions (if any) to the equation $f(x) = -2x^2 - 8x - 8 = 0$? What is the factored form of f?

$$D = b^2 - 4ac = (-8)^2 - 4(-2)(-8) = 64 - 64 = 0$$

$$x_1 = -\frac{b}{2a} = -\frac{(-8)}{2(-2)} = -2$$

So $f(x) = a(x - x_1)^2 = -2(x+2)^2$

- What are the solutions (if any) to the equation $f(x) = -x^2 + x - 6 = 0$? What is the factored form of f?

$$D = b^2 - 4ac = (1)^2 - 4(-1)(-6) = 1 - 24 = -23$$

$\rightarrow$ no solutions

$\rightarrow$ $f(x)$ cannot be factored.

We can now also use this technique to solve the equation associated with our case study earlier, in a different way:

$$f(p) = -200p^2 + 1200p - 1500 = 0$$

$$D = b^2 - 4ac = (1200)^2 - 4(-200)(-1500) = 1440000 - 1200000$$
$$= 240000$$

$$p_{min,max} = \frac{-1200 \pm \sqrt{240000}}{2(-200)} = \frac{-12 \pm \sqrt{24}}{-4} = 3 \pm \frac{\sqrt{4 \times 6}}{-4}$$
$$= 3 \pm \frac{\sqrt{3}}{\sqrt{2}} \text{ as before}$$

Finally, it is worth noting that this method can also help solve a few higher-order equations that can be reduced to a quadratic, as in these examples:

- What are the solutions (if any) to the equation $f(x) = x^6 - 3x^3 - 9 = 0$?

let $u = x^3$ then $\cancel{f}$ $x^6 - 3x^3 - 9 = 0 \iff u^2 - 3u - 9 = 0$

$$D = b^2 - 4ac = (-3)^2 - 4(1)(-9) = 9 + 36 = 45$$

$$\rightarrow u_{1,2} = \frac{-(-3) \pm \sqrt{45}}{2} = \frac{3 \pm 3\sqrt{5}}{2} = x_{1,2}^3$$

then $x_{1,2} = \sqrt[3]{\dfrac{3 \pm 3\sqrt{5}}{2}}$

- What are the solutions (if any) to the equation $f(x) = x^4 - 2x^2 - 3 = 0$?

let $u = x^2$ then $x^4 - 2x^2 - 3 = 0 \iff u^2 - 2u - 3 = 0$

$$D = b^2 - 4ac = (-2)^2 - 4(1)(-3) = 4 + 12 = 16$$

$$u_{1,2} = \frac{-(-2) \pm \sqrt{16}}{2} = \frac{2 \pm 4}{2} = \begin{cases} 3 \\ -1 \end{cases} = x_{1,2}^2$$

so $x_1^2 = 3 \Rightarrow x_1 = \pm\sqrt{3}$

$x_2^2 = -1$ does not have solutions

So the solutions are $x = \pm\sqrt{3}$

GRAPHING QUADRATICS. Let's now recap everything we know about the graphs of quadratic functions based on their mathematical expression. Given $f(x) = ax^2 + bx + c$,

- The graph is a parabola
- It opens up if $a>0$, down if $a<0$
- The vertex is at $x_v = -\frac{b}{2a}$, $y_v = f(x_v)$
- The tangent at $x=0$ is $y = bx+c$
- Given $D = b^2 - 4ac$:
  - if $D<0$ there are no roots (no x-intercept)
  - if $D=0$ then the parabola has 1 root at the vertex (whose coordinates are $x_v = -\frac{b}{2a}$, $y_v = 0$)
  - if $D>0$ the parabola has 2 roots, at
    $x_1 = \frac{-b+\sqrt{D}}{2a}$ $\quad x_2 = \frac{-b-\sqrt{D}}{2a}$

Based on all this information (much of which is actually redundant) we can easily graph the parabola.

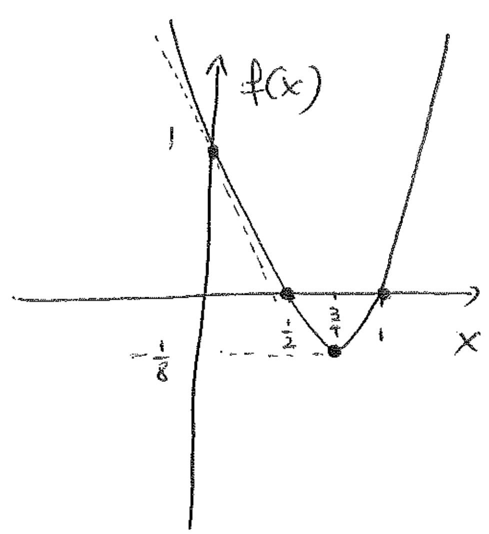
EXAMPLE 1: $f(x) = 2x^2 - 3x + 1$

roots are $x_{1,2} = \frac{+3\pm 1}{4} = \begin{cases} +1 \\ +\frac{1}{2} \end{cases}$

- opens up
- tangent $y = -3x+1$
- vertex: $x_v = \frac{-(-3)}{2(2)} = \frac{3}{4}$

$$y_v = 2\left(\frac{3}{4}\right)^2 - 3\left(\frac{3}{4}\right) + 1$$
$$= 2\cdot\frac{9}{16} - \frac{9}{4} + 1$$
$$= -\frac{1}{8}$$

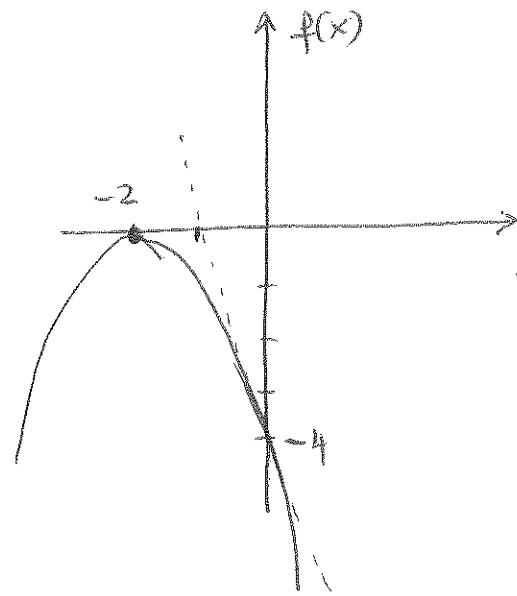
- $D = (-3)^2 - 4(2)(1) = 9-8 = 1$



EXAMPLE 2: $f(x) = -x^2 - 4x - 4$

- opens down
- tangent: $y = -4x - 4$
- vertex: $x_v = \frac{-(-4)}{2(-1)} = -2$
- $y_v = -(-2)^2 - 4(-2) - 4 = -4 + 8 - 4 = 0$
- $D = (-4)^2 - 4(-1)(-4) = 16 - 16 = 0$

  $\rightarrow$ one root, at vertex



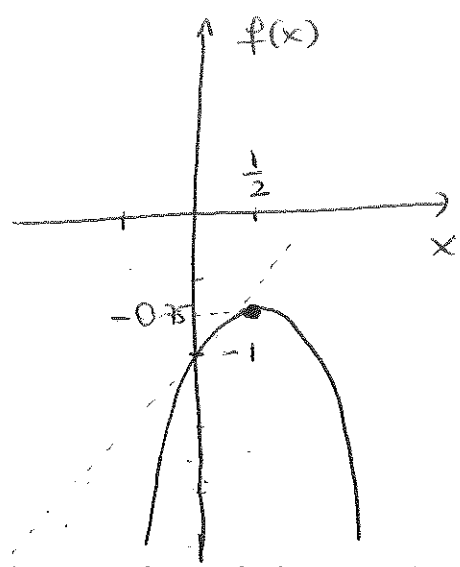
EXAMPLE 3: $f(x) = -x^2 + x - 1$

- opens down
- tangent: $y = x - 1$
- vertex: $x_v = \frac{-(1)}{2(-1)} = +\frac{1}{2}$

$$y_v = -\left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right) - 1 = -\tfrac{1}{4} + \tfrac{1}{2} - 1 = -0.75$$

$$D = (1)^2 - 4(-1)(-1) = 1 - 4 = -3$$

$\rightarrow$ no roots



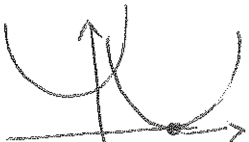
SIGN OF A QUADRATIC: As we saw at the beginning of this Section on quadratics, for large enough x (both positive and negative)

$$f(x) = ax^2 + bx + c \simeq ax^2 \text{ for large } x$$

Since x^2 is always positive, this means that, for large enough x

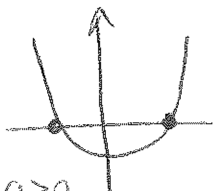
$f(x)$ has the same sign as a for large enough x

With this in mind, we can now figure out what the sign of $f(x)$ is for all values of x, for all three possible scenarios. Indeed, the only way $f(x)$ can change sign is when the parabola actually crosses the $x-$axis, that is, when $f(x)$ has two different roots.



a>0
example

- If $f(x)$ has 0 or 1 root, then $f(x)$ does not change sign, and <u>always</u> has the same sign as a.



a>0
example

- If $f(x)$ has 2 roots, then $f(x)$ has the sign of a outside the roots, and minus the sign of a inside the roots

This is quite useful when trying to find, for instance, the domain of definition of functions defined as the square roots of quadratics.

- $f(x) = \sqrt{x^2 - x + 6}$ $\quad D = b^2 - 4ac = (-1)^2 - 4(1)(6) = -23$

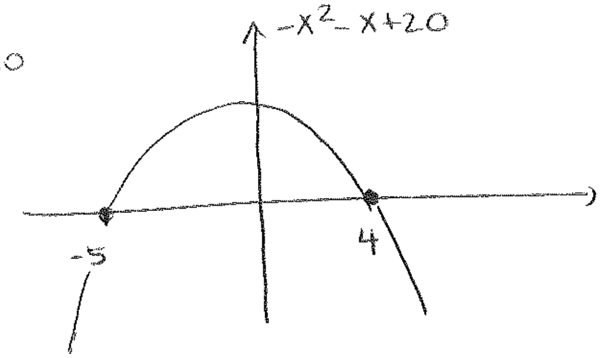
  → no roots, so $x^2 - x + 6$ has the same sign as $a = 1$ everywhere ⇒ $x^2 - x + 6$ is always > 0

  → $\mathcal{D} = \mathbb{R} = (-\infty, +\infty)$

- $f(x) = \sqrt{-x^2 - x + 20}$ $\quad D = b^2 - 4ac = (-1)^2 - 4(-1)(20) = 1 + 80 = 81$

  $$x_{1,2} = \frac{-(-1) \pm \sqrt{81}}{2(-1)} = \frac{1 \pm 9}{-2} = \begin{cases} -5 \\ 4 \end{cases}$$

so



→ is positive within the roots, in $[-5, 4]$

→ is negative outside the roots in $(-\infty, -5] \cup [4, +\infty)$

as predicted.

→ $\mathcal{D} = [-5, 4]$