Chapter 2

Polynomial functions

In this Chapter we will study a large class of functions called "polynomial functions", which all take the form:

The simplest example of these polynomial functions are the linear functions.

2.1 Linear functions

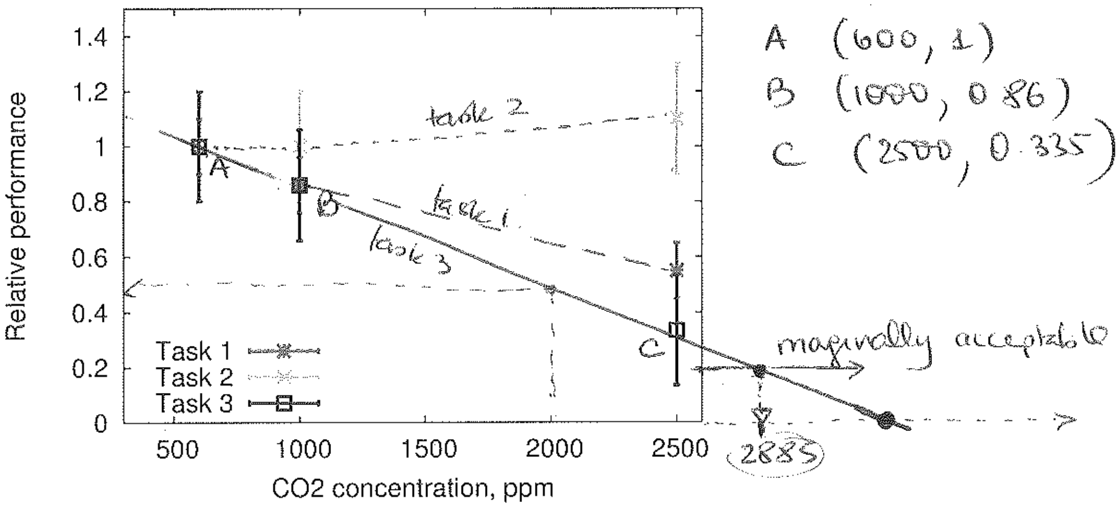
Textbook Section 2.1 and 2.2.

2.1.1 Case study: CO_2 as an indoor pollutant

This case study is based on the findings of a paper entitled "Is CO_2 an indoor pollutant? Direct effects of low to moderate CO_2 concentrations on human decision-making performance.", by a team of researchers from Upstate Medical University (Syracuse, NY) and from Lawrence Berkeley National Labs, in 2012 (see website for a link). Researchers asked 22 volunteers to take part in computerized decision-making tests that each take 2.5 hours, while being in an environment containing more CO_2 than normal, at levels of 600 ppm (considered normal for indoor environments), 1000ppm (slightly above normal), and 2500ppm (significantly above normal). None of these levels are thought to be have long term harmful effects to health, but they were demonstrated in this experiment to have statistically significant impacts on various aspects of the decision making processes. Nine kinds of tasks were tested, ranging from basic activities, to information search and usage, to basic use of strategy, using a well-known and well calibrated decision-making test. The following table summarizes test results for 3 of these tasks, scaled to their "normal value" at a CO_2 concentration of 600ppm (so by construction, the scaled results is equal to one in that case).

CO_2	Task 1	Task 2	Task 3
600 ppm	1 ± 0.1	1 ± 0.2	1 ± 0.2
1000 ppm	0.86 ± 0.1	1 ± 0.2	0.86 ± 0.2
2500 ppm	0.55 ± 0.1	1.1 ± 0.2	0.335 ± 0.2

The errors are related to the spread in the test scores of each individual participants, while the main value reported is the average of all the test scores. The same information is presented in the following graph:



From this graph, we can immediately see a few important things:

- For Task 1 & 3, the test scores decrease as the CO_2 level in the room increases.
- For Task 2, the test scores increase a little bit.
- For each task the data points lie close to a straight line.

However, can we be more quantitative about some of these statements? And can we use this information to predict, what the relative decision-making performance of someone may be as a function of the ambient CO_2 concentration for any possible input values of the latter? To do this, we need to create a function that takes as input the value of the ambient CO_2 concentration, and returns the relative test score. With such knowledge, one would be able to better predict the effects of CO_2 on the performance of workers in a company, or of students in test-taking environments, depending on the tasks they have to perform. Let us now learn a little bit more about linear functions, which we shall see are quite relevant for this case study.

2.1.2 Mathematical corner: Definition and basic properties of linear functions

DEFINITION: A linear function can be written as $f(x) = mx + b$ (m and b are real parameters)

examples: $f(x) = x + 1$ $\quad f(x) = -3x - 4$

$g(y) = 5y + 2$ $\quad g(y) = 10y$ $\quad g(y) = 4$

$h(z) = \sqrt{2}\,z + \sqrt{3}$ $\quad P_{\text{instore}}(n) = 16n$

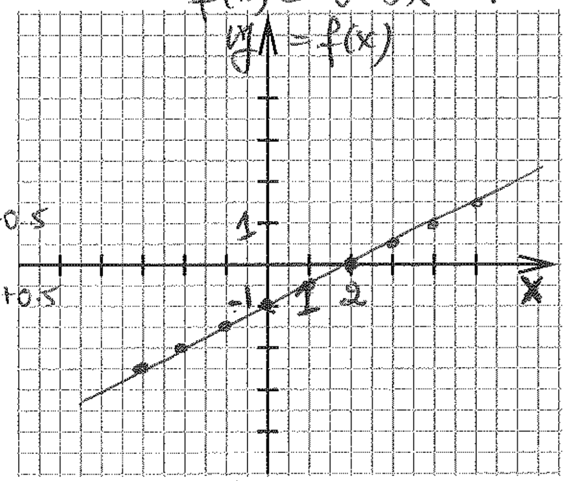
DOMAIN OF DEFINITION OF A LINEAR FUNCTION: A linear function is defined for all real numbers; $\mathcal{D} = \mathbb{R}$

EXAMPLES OF LINEAR FUNCTIONS AND THEIR GRAPHS:

x	$0.5x-1$
-3	-2.5
-2	-2
-1	-1.5
0	-1
1	-0.5
2	0
3	0.5

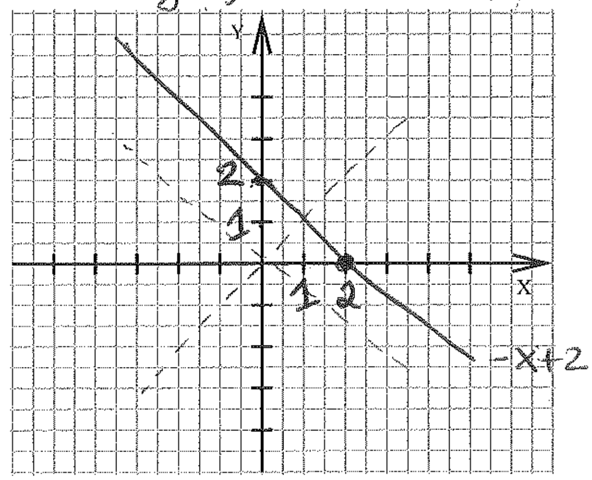
$+0.5$ $+0.5$

$f(x) = 0.5x - 1$



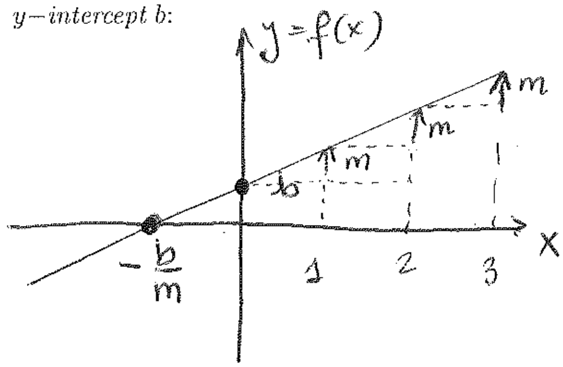
A straight line with slope 0.5 (0.5 up for each 1 to the right)

$g(x) = 2 - x = -x + 2$



A straight line with slope -1 (1 down for each 1 to the right)

GRAPH OF A LINEAR FUNCTION: The graph of $y = f(x) = mx + b$ is a straight line with *slope* m and y*−intercept* b:



The y-intercept is where the graph $y = f(x)$ intersects the y-axis. For a linear function $f(0) = m \cdot 0 + b = b$
$\rightarrow$ y intercept is $(0, b)$

The slope is defined as the change in y divided by change in x. If the change in x is 1 then

$$\text{slope} = \frac{f(x+1) - f(x)}{1} = \frac{[m(x+1)+b] - [mx+b]}{1}$$
$$= \cancel{mx} + m + \cancel{b} - \cancel{mx} - \cancel{b} = m$$

Note that we can also define the x*−intercept*:

The x-intercept is where the graph of $y = f(x)$ intersects the x-axis. (ie, when $y = 0$). To find it we solve the equation $f(x) = 0$ for x:

$$mx + b = 0 \longrightarrow mx = -b \longrightarrow \boxed{x = -\frac{b}{m}}$$

EXAMPLES:

- $f(x) = 0.5x - 1$ ($mx + b$) y-intercept is -1 slope is $\frac{1}{2}$
  x-intercept is $x = \frac{-(-1)}{0.5} = 2$

- $f(x) = 2 - x = -x + 2$ y-intercept : 2 slope : -1
  x-intercept : $x = \frac{-2}{-1} = 2$

BASIC PROPERTIES OF A LINEAR FUNCTION:

- A linear function is continuous (graph can be drawn without lifting pen)
- A linear function is increasing if $m > 0$
- " " " is decreasing if $m < 0$
- A linear function is constant if $m = 0$ (graph is a horizontal line).

2.1.3 Case study: CO_2 as an indoor pollutant (part 2)

Going back to our case study, we want to find three linear functions that model how in the test performance depends on the CO_2 abundance for each of the three tasks. Let us look at Task 3 first. In order to find a linear function to model the data, we would (in principle) like to find two numbers m and b such that

→ we want to find a function $f(x) = mx + b$ where x = input CO_2 concentration, $f(x)$ = output score, such that

$$\begin{aligned} f(600) &= 1 &\to \quad & m \cdot 600 + b = 1 \\ f(1000) &= 0.86 &\to \quad & m \cdot 1000 + b = 0.86 \\ f(2500) &= 0.335 &\to \quad & m \cdot 2500 + b = 0.335 \end{aligned}$$

Recalling that m is the slope of the graph, we can find m by calculating the slope between any pair created from these three points.

We know that m is the slope of the line we want to fit → that's given by $m = \frac{\text{change in } y}{\text{change in } x}$

$$m = \frac{y_B - y_A}{x_B - x_A} = \frac{0.86 - 1}{1000 - 600} = -0.00035$$

$$m = \frac{y_C - y_A}{x_C - x_A} = \frac{0.335 - 1}{2500 - 600} = -0.00035 \quad ✓$$

Note that the slope is negative as expected from data

It should not be surprising that we find the same answer in each case : having a constant slope is what defines a straight line! In fact, had we found different answers, this would have been an indication that the points do not lie in a straight line (see below for an example).

To find b knowing m, we then have to solve for it using one (or all) of the three equations above.

Use for instance $m \cdot 600 + b = 1$, with $m = -0{,}00035$

$-0.00035 \cdot 600 + b = 1 \longrightarrow -0.21 + b = 1$

$\rightarrow b = 1 + 0.21 = 1.21$

Note: we could have used $m \cdot 1000 + b = 0.86$ instead

$-0.00035 \cdot 1000 + b = 0.86 \Rightarrow -0.35 + b = 0.86$

$\rightarrow b = 0.86 + 0.35 = 1.21$

Again we expect to find the same answer in each case – had this not been the case, this would have been a hint that the points are not in a straight line. Now that we know m and b, we can finally write that the function describing the relationship between relative test performance and CO_2 concentration is

$$\rightarrow \quad f(x) = -0.00035\,x + 1.21$$

For instance we can now predict what the relative test score would be for say, CO_2 concentration of 2000 ppm: $f(2000) = 0.51$

With some interpretation of the results of Figure 2, we can state that any relative score below 0.2 for this task is considered one in which the decisions made by the test subjects are only marginally acceptable. Above which value of the CO_2 concentration does his happen? We can answer this equation either by looking at the graph, or mathematically.

$\rightarrow$ Solve $f(x) = 0.2$ for $x \Rightarrow 0.2 = -0.00035x + 1.21$

$\rightarrow 0.2 - 1.21 = -0.00035x \rightarrow -1.01 = -0.00035x$

$$\rightarrow \quad x = \frac{-1.01}{-0.00035} \cong 2885 \text{ ppm}$$

Similarly, any relative test score below 0 indicates systematically dysfunctional decision making. Above which CO_2 concentration does this happen? Again, we can answer this equation either by looking at the graph, or mathematically.

Solve $f(x) = 0$ for x → that's the same as finding the x-intercept. $\left(x = -\frac{b}{m}\right)$

$-0.00035x + 1.21 = 0$

$$x = \frac{-1.21}{-0.00035} \cong 3457 \text{ ppm}$$

Let's now look at Tasks 1 and 2. We clearly see from their graphs that the data does not lie in a straight line, which means that

The data cannot strictly be fitted with a linear function.

However, we also see that if we draw a straight line between the first and the last point, that straight line safely lies within the error bars for the middle point. For Tasks 1 and 2-Body, therefore, we will only use the first and the last point to create the linear function. Let's apply the same steps as before:

To summarize, we have the following formulas for each of the three functions:

-
-
-

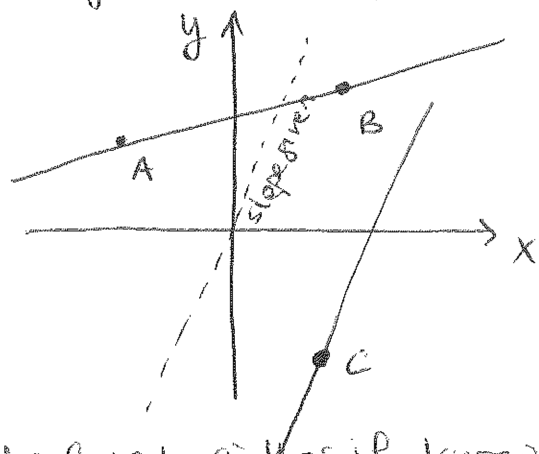
We see that

2.1.4 Mathematical corner: Properties of lines (a little bit of geometry)

Studying linear functions is easier if we remember a little bit about properties of lines, since the graph of a linear function *is* a line.

Geometrically speaking, a line is *uniquely* defined either by

- Two points (i.e, the line that goes through 2 points)
- A point and a slope (i.e, the line that goes through the point, with given slope)



CONSEQUENCE:

The equation of a line can be found either if know two points OR if we know a point and a slope

Note that if you know the coordinates of the two points, you can calculate the slope of the line going through the points: $A(x_A, y_A)$ $B(x_B, y_B)$

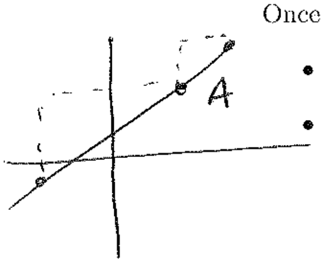
$$\text{slope} = \frac{\text{change in } y}{\text{change in } x} = \frac{y_A - y_B}{x_A - x_B} = \frac{y_B - y_A}{x_B - x_A}$$

Also, remember that

- Two lines with the same slope are parallel.
- A line parallel to the x-axis has slope 0 and its equation is $y = b$
- (A line parallel to the y-axis is ~~the~~ not the graph of a function. Its equation is $x = c$ (c constant))

Once the slope of a line is known, there are two ways of writing the line equation:

- The slope-intercept formula (if you know the y−intercept): $\boxed{y = mx + b}$ m: slope, b: y-intercept
- The point-slope formula (if you know a point on the line):



$$y - y_A = m(x - x_A) \quad \left(\text{equivalently, } m = \frac{y - y_A}{x - x_A}\right)$$

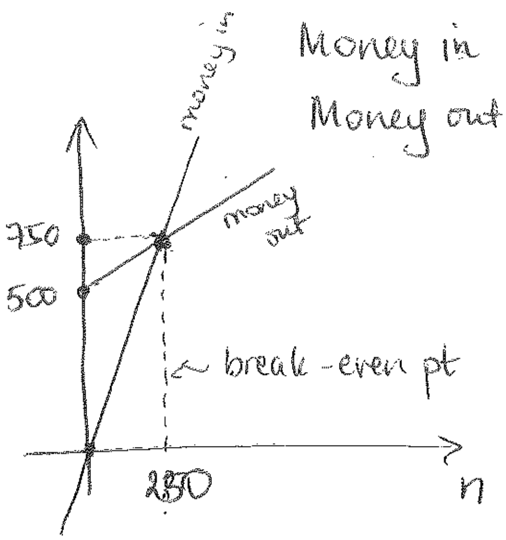
2.1.5 Case study: Cost and revenue: how to decide whether a market is viable

Donnelly's in Santa Cruz sell some of the finest chocolates in the US. The chocolates are sold at \$3 a piece. The rental of the facility costs \$200 per day, the salaries & benefits of the owners and workers costs \$300/day and the raw material for each chocolate costs \$1 per piece. How many chocolates do they have to sell per day to break even? What will be the amount of money involved (in and out) at that point?

let n be the number of chocolates sold per day

$$\text{Money in}(n) = 3n$$

$$\text{Money out}(n) = 200 + 300 + n = 500 + n$$



money in = money out

$$3n = 500 + n \quad \leftarrow \text{an equation for } n$$

$$3n - n = 500$$

$$2n = 500 \rightarrow \boxed{n = 250}$$

At this point

$$\text{money in} = \text{money out} = 3 \times 250 = 500 + 250 = 750$$

2.1.6 Mathematical corner: Solutions of linear equations

In many real life applications, we may be led to solve a system of linear equations, i.e. two linear equations for two unknowns. We just saw one example:

$$\text{money}_{in}(n) = 3n \qquad \text{money}_{out}(n) = 500 + n \rightarrow \text{solve money}_{in} = \text{money}_{out}$$

However, more generally, we may have to solve a set of equations of the kind

$$y = ax + b \qquad y = cx + d \quad \leftrightarrow \quad \text{when do the lines intersect?}$$

To do so, simply note that y is the same in both equations, so we can set them equal to one another and solve for x. We effectively eliminate one of the unknowns by doing that, and just solve for the other. We get

$$y = \underbrace{ax + b = cx + d}_{\text{solve for } x}$$

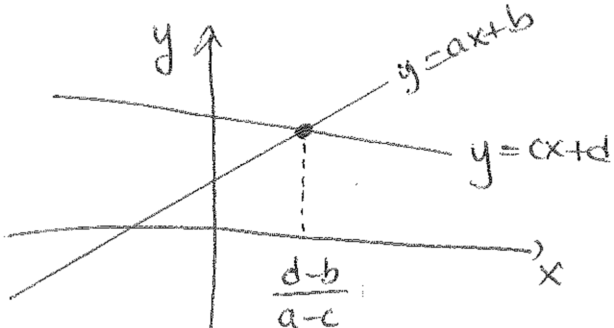
$$ax + b = cx + d$$

$$ax - cx = d - b$$

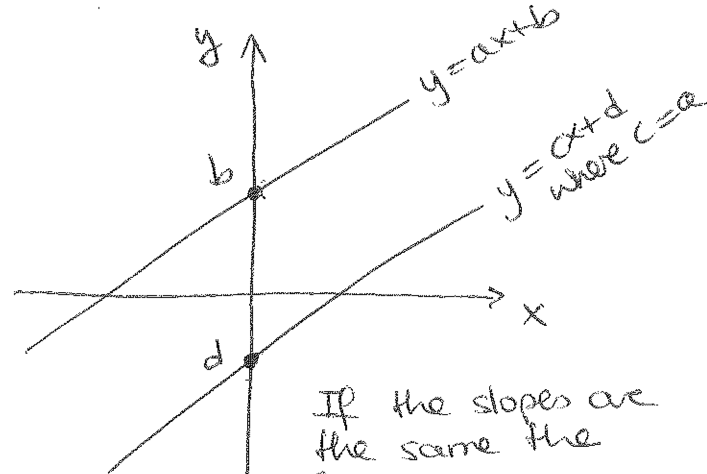
$$x(a - c) = d - b$$

$$\boxed{\text{If } a - c \neq 0} \qquad \boxed{x = \frac{d - b}{a - c}}$$

It's interesting to note that if $a - c = 0$, then there is a problem! To see why, let's interpret this system of equations geometrically.



If the slopes are $\neq$ the lines intersect (always).



If the slopes are the same the lines cannot cross (at best they are the same line)

We see that

- If $a \neq c$ then there is no problem and there is a single solution to the system of equations
- If $a = c$ then either
  - $b \neq d$ then there is NO solution to the problem
  - $b = d$ then any x is a solution

Once the solution for x is known (assuming it exists and it is unique) we can use it to find y. Note that we can use either of the two original equations to do so, and get the same result:

$$y = ax + b = a\frac{d-b}{a-c} + b = \frac{ad - \cancel{ab} + \cancel{ba} - bc}{a-c} = \frac{ad-bc}{a-c}$$

$$y = cx + d = c\frac{d-b}{a-c} + d = \frac{\cancel{cd} - cb + da - \cancel{dc}}{a-c} = \frac{ad-bc}{a-c} \checkmark$$

Even more generally, we may have to solve coupled systems of the kind

$$Ax + By = C \qquad Dx + Ey = F$$

Note that this last set can be reduced to the one before by solving each equation for y first, and then following the same steps as before.

$$Ax + By = C \longrightarrow y = \frac{C - Ax}{B} = \frac{C}{B} - \frac{A}{B}x$$

$$Dx + Ey = F \longrightarrow y = \frac{F - Dx}{E} = \frac{F}{E} - \frac{D}{E}x.$$

EXAMPLES:

- Solve the system of equations $y = 3x + 1$, and $y = -2x - 4$ for x and y

$$y = \underbrace{3x+1 = -2x-4}$$

$$3x + 2x = -4 - 1 \rightarrow 5x = -5 \Rightarrow x = -1$$

$$y = 3(-1) + 1 = -2 \quad \text{or Check: } y = -2(-1) - 4$$
$$= 2 - 4 = -2 \checkmark$$

- Solve the system of equations $y = x + 1$, and $y = x + 4$

- Solve the system of equations $2x + 3y = 1$, and $x - 3y = 2$

- Solve the system of equations $2x + 4y = 1$, and $x + 2y = \frac{1}{2}$