Chapter 1

The notion of functions

Textbook Chapter 1

1.1 The concept of functions

Although the concept of functions was invented a very long time ago, it is very easy today to gain an intuitive notion of what functions are because of their natural role in most computer and/or web-based applications, in engineering, and in economics, etc. Functions can also be used to model data that has been collected, and provide a simple way of sharing and analyzing this data. We will now look at two case-studies that showcase the use of functions in both contexts.

1.1.1 Case Study: *Online vs. in-store purchase of bulk items*

When buying bulk items online, the price is often somewhat cheaper than in store, but one usually needs to add an extra fee for shipping. It pays off to do a quick little calculation to find out which ends up being the cheaper of the two options, depending on the quantity purchased. Consider for instance buying diapers:

- *In store, a pack of 30 diapers usually sells for about $16.*
- *Online, the same pack usually sells for $14, but there is a $10 shipping fee to add on the total order (regardless of the number of packs bought).*

Let's make a table to compare the prices in each as, as a function of the number of packs n bought:

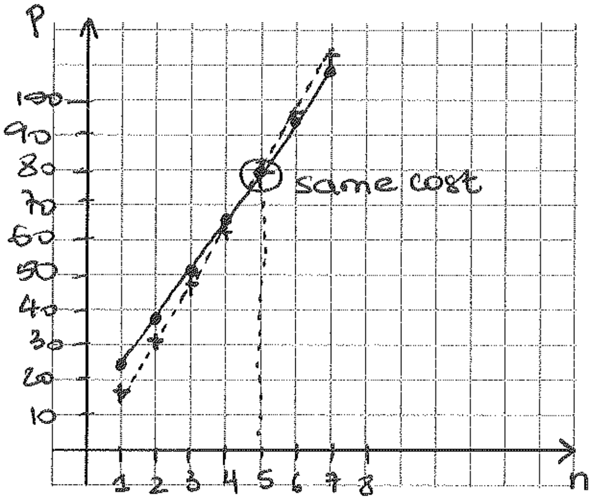
n	1	2	3	4	5	6	7	Formula
Online	24	38	52	66	80	94	108	$10+14n$
In store	16	32	48	64	80	96	112	$16n$

We see that each price can be obtained automatically knowing n simply by applying a simple formula in each case:

- $P_{online}(n) = 10 + 14n$
- $P_{store}(n) = 16n$

Furthermore, we see from the table that buying online becomes cost-effective when: $n \geq 5$

If we graph of the price of the purchase vs. the number of packs bought, we arrive at the same conclusion: the graph carries the same information as the table, but in a way that is immediately apparent to the eye.



- The graph of total cost vs. number of packs is a straight line in both cases
- The line for online price is <u>above</u> line for in-store price for $n<5$, and <u>below</u> for $n>5$.

$\Rightarrow$ in store is more cost effective for $n<5$ but online is better for $n>5$.

Using the same technique, we can in principle find the minimum purchase amount to be cost effective for any item we may be interested in, provided we know the in-store price, the online price, and the shipping fees. However, creating a table or a graph every time can be a bit time-consuming. Later in this lecture we will do this in a much more efficient way.

1.1.2 Mathematical corner: Functions defined by a mathematical rule

DEFINITION: A function defined by a <u>mathematical rule</u> takes one number, applies a formula, and returns the result. We usually write $x \longrightarrow f(x) =$ some formula

input (the x); name of function (the f)

EXAMPLES:

- $x \longrightarrow f(x) = x+1$: takes any number, returns number + 1
- $x \longrightarrow f(x) = |x|$: takes a number, returns its absolute value
- $y \longrightarrow g(y) = y^2 - 1$
- $z \longrightarrow h(z) = a + z^2$

Note: • The name of the function does not have to be f • The name of variable does not have to be x

- $n \longrightarrow P_{store}(n) = 16n$; $P_{online}(n) = 10 + 14n$.

DEFINITIONS: If a quantity y is related to a quantity x via $y = f(x)$ we say that x is the <u>independent variable</u> and y is the <u>dependent variable</u>. Any other letters or variables in the expression, if they are present, are called <u>parameters</u>

EXAMPLES:

- $y = f(x) = x + 1$: y = dependent, x = independent
- $C = p(n) = 16n$ C = dependent, n = independent
- $y = h(z) = a + z^2$: y = dependent, z = independent
  a = a parameter.

To evaluate a mathematical function for any input value of the independent variable, simply replace it in the mathematical formula by the desired number.

EXAMPLES: To find the cost of 10 diaper bags online and in-store, we calculate:

- $P_{store}(10) = 16 \times 10 = 160$
- $P_{online}(10) = 10 + 14 \times 10 = 150$

We also have

$$f(x) = x + 1 \Rightarrow f(\pi) = \pi + 1$$

$$h(z) = a + z^2 \Rightarrow h(1) = a + 1^2 = a + 1$$

$$g(y) = y^2 - 2 \Rightarrow g(2) = 2^2 - 2 = 4 - 2 = 2$$

A function can also be applied to other variables and even whole expressions instead of numbers. In that case, simply replace the dependent variable by the whole expression. Although this may seem strange at first, we will see later why this can be very handy.

EXAMPLES:

- $f(x) = x + 1 \Rightarrow f(x^2) = x^2 + 1$
- $h(z) = a + z^2 \Rightarrow h(z-1) = a + (z-1)^2$
- $g(y) = y^2 - 2 \Rightarrow g(x+2) = (x+2)^2 - 2$

Domain of definition of a function: The *Domain of Definition* of a function f consists of all of the values x for which we **are allowed to** or **want to** assign a value $y = f(x)$.

- "**allowed to**" refers to the mathematical rules, i.e. when are you allowed to apply that rule to x

  Examples:

  1. $f(x) = \frac{1}{x-1}$

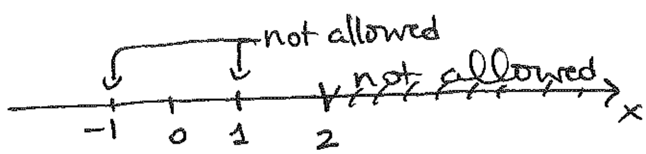
     Denominator cannot be zero $\Rightarrow x-1 \neq 0$
     $\Rightarrow x \neq 1$. $\mathcal{D} = \{x \neq 1\}$ or $\mathcal{D} = \mathbb{R} - \{1\}$
     or $\mathcal{D} = (-\infty, 1) \cup (1, +\infty)$

  2. $f(x) = \sqrt{x-1}$

     Radical cannot be negative $\Rightarrow x-1 \geq 0 \Rightarrow x \geq 1$
     so $\mathcal{D} = \{x \geq 1\}$ or $\mathcal{D} = [1, +\infty)$

  3. $f(x) = \frac{\sqrt{2-x}}{x^2-1}$

     • Radical cannot be negative $\Rightarrow 2-x \geq 0 \Rightarrow 2 \geq x$
     AND • x^2-1 cannot be zero $\Rightarrow x^2 \neq 1 \Rightarrow x \neq 1$ and $x \neq -1$

     $\Rightarrow$ need $\boxed{x \leq 2 \text{ and } x \neq 1 \text{ and } x \neq -1}$

- "**want to**" refers to the physical problem considered, i.e. what are the values of x that make sense?

  Example : What is the domain of definition of the price of diapers?

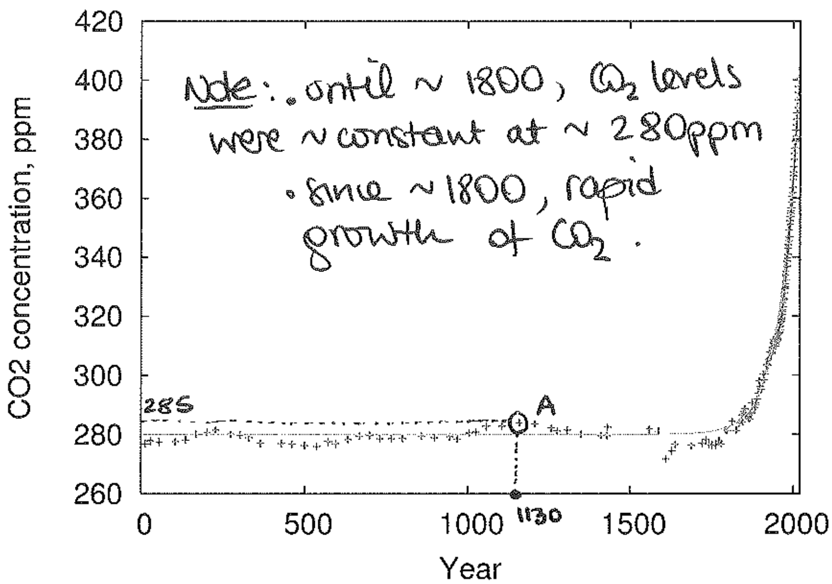
  There is no point "buying zero packs" or "buying a negative number of packs": so for $P_{store}(n)$ and $P_{online}(n)$, $\mathcal{D} = \{n > 0\}$

1.2 Functions and graphs

As we saw in the previous section, graphs are an easy way to *visualize* a mathematical function so that it is easier to interpret. We will learn more about this in this section. Conversedly, when graphs are obtained purely from data, one may be interested in finding out what function best models this data. In some cases, a well-chosen fit to the data can be used to summarize the data and share it more easily with others (ie. by sharing the function rather than each data point). Sometimes, this can even lead us to discover interesting things about the data that may not have been immediately apparent.

1.2.1 Case Study: *CO_2 concentration in the atmosphere*

One of the most famous graphs of the last 2 decades depicts the concentration of CO_2 in the atmosphere as a function of time. Each point on this graph has the measurement year as its $x-$coordinate, and the measured CO_2 concentration as its y-coordinate. Note that early measurements (pre 1958) come from estimates of the CO_2 abundance derived from analysis of ice-cores, while the post 1958 measurements are direct measurements taken from Mauna Loa (HI). This graph immediately shows:



Point A has coordinates (1130, 285)

Figure 1.1: CO_2 concentration as as function of time. Data from Scripps.

The green line is a function fitted to the data with a little trial and error:

$$C = f(y) = 280 + e^{0.018(y-1750)}$$

$C = CO_2$ abundance y = year

We see that it fits the data reasonably well from year 0 to today. This can be useful, since we can now use this formula to get a rough estimate of what the CO_2 concentration was historically at any time since year 0. For instance,

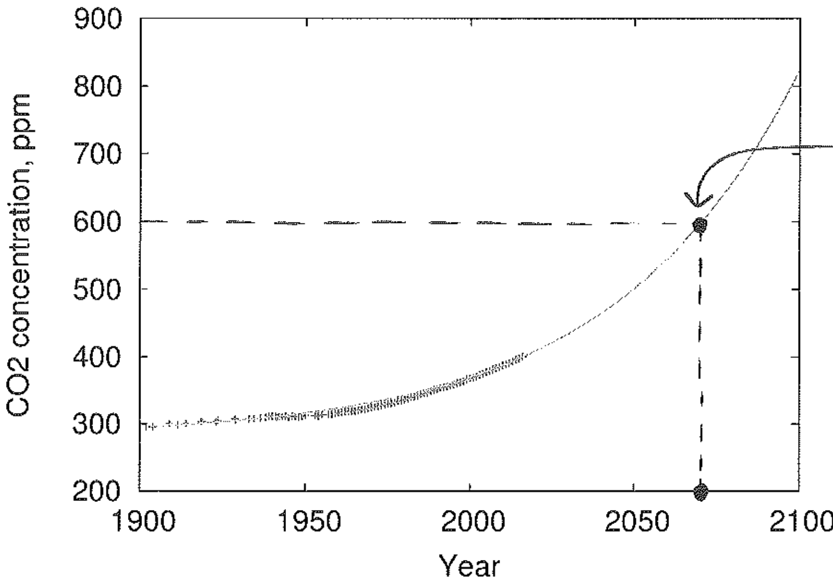
- *In 1980:* $C = CO_2 \text{ abundance} = f(1980) = 280 + e^{0.018(1980-1750)} \cong 342.8$
- *In 2000:* $C = f(2000) \cong 280 + e^{0.018(2000-1750)} \simeq 370$

We can also share this information quite easily with other people, simply by sharing the formula, rather than the entire dataset.

In addition, if we assume that the CO_2 abundance will continue to grow at this rate (i.e. following the same function), we can estimate how high it will be in 2020 and 2030 for instance:

$$f(2020) = 280 + e^{0.018(2020-1750)} \cong 409$$
$$f(2030) = 280 + e^{0.018(2030-1750)} \simeq 434$$

Again, assuming that the CO_2 abundance continues to grow at this rate, we can also estimate how long it will take to reach 600 ppm, which is thought by many scientists to be one of the tipping points for the Earth's climate. To do so, we look at the following graph:



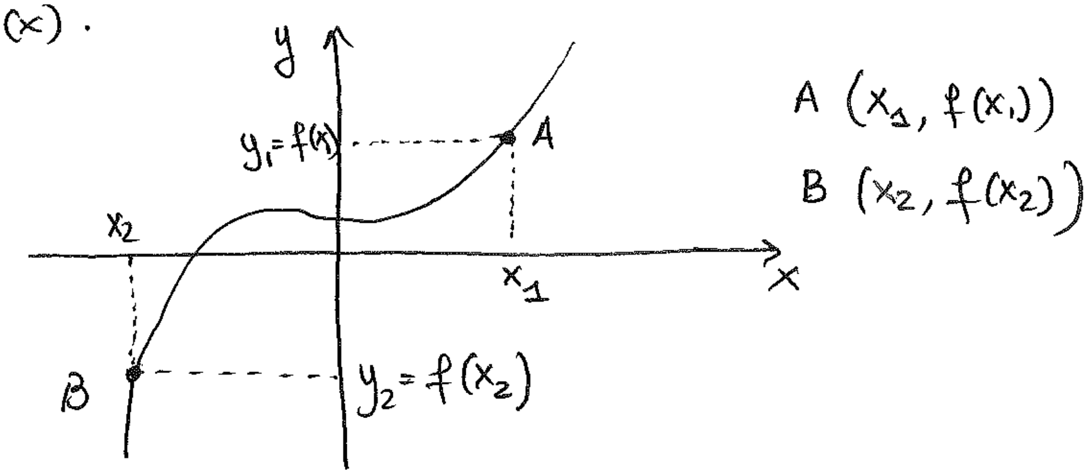
The point at $y = 600$ ppm has x-coordinate $\simeq 2070$

$\Rightarrow$ At present rate, the "tipping point" will be reached by ~ 2070.

Note that nothing guarantees that the actual CO_2 concentration will actually continue to follow this trend in the next 50 years. This is merely an empirical model, and should not be viewed as a real prediction. In reality, the growth may continue to accelerate even more, or may begin to slow down, depending on the political decisions we make and the response of the global Carbon cycle to this dramatic change. If that is the case a different modeling function would be required to get a better prediction.

1.2.2 Mathematical corner: Graphs of standard functions and graphing techniques

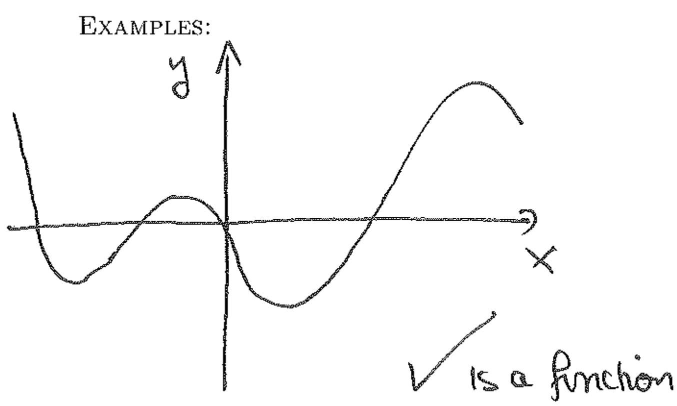
DEFINITION: The graph of a function $f(x)$ is the set of all points with coordinates (x, y), where $y = f(x)$.

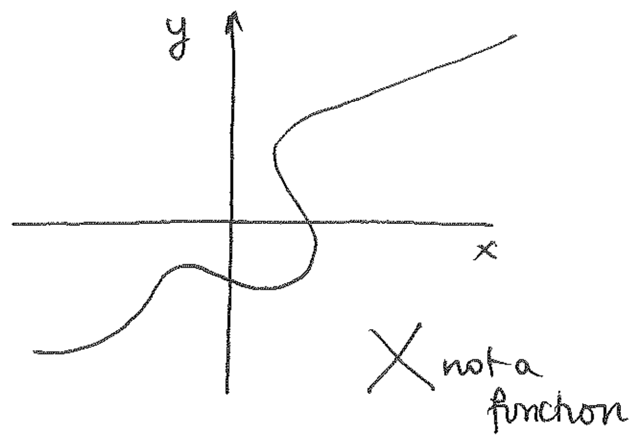


Single Value Property:

- Saying "$y = f(x)$" only makes sense when there is a single value of y corresponding to each value of x. This means that while every function has a graph, not every graph can be the graph of a function!
- The vertical line test: A *graph* corresponds to a *function* only if it passes the *Vertical Line Test*: if any vertical line on the graph intersects the line $y = f(x)$ more than once, then f is not Single Valued, therefore f is not a function.

Examples:





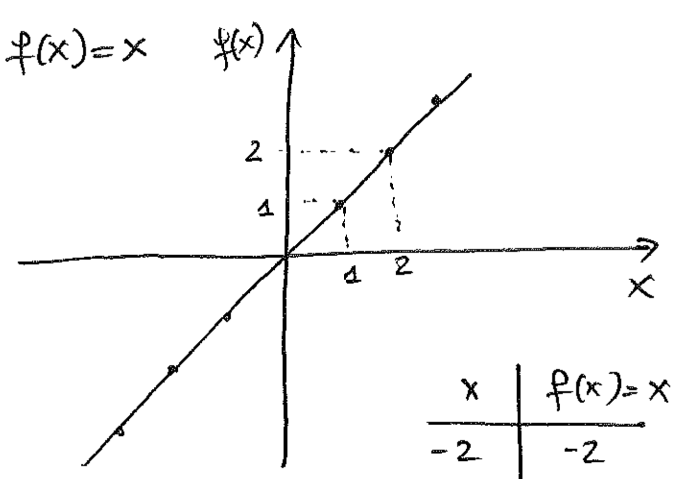
The most basic graphing technique is

- to construct a table of values, with two columns: values of x and corresponding values of $f(x)$
- draw the corresponding points with coordinates $(x, f(x))$.

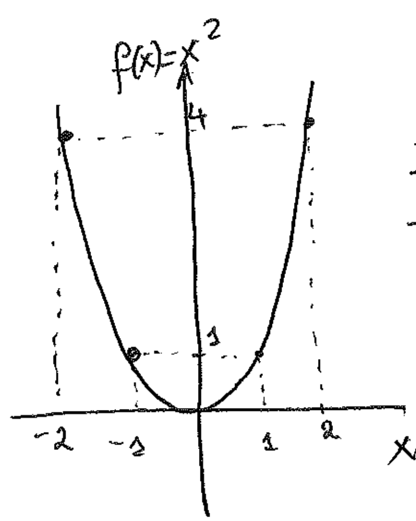
We have seen a few examples of this technique already. However, graphing point by point be a little bit tedious, and there are a lot of faster techniques. They are based on knowing the graphs of "standard" functions, and how mathematical and geometrical manipulations of these graphs relate to one another.

The standard functions

The following standard functions are ones everyone should know how to graph, accurately, "by heart", from now on. As the course proceeds, we will continue adding to this list of functions.



x	f(x)=x
-2	-2
-1	-1
0	0
1	1
2	2

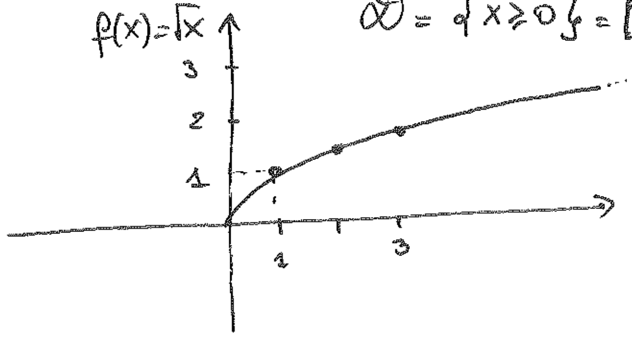


x	f(x)
-2	4
-1	1
0	0
1	1
2	4

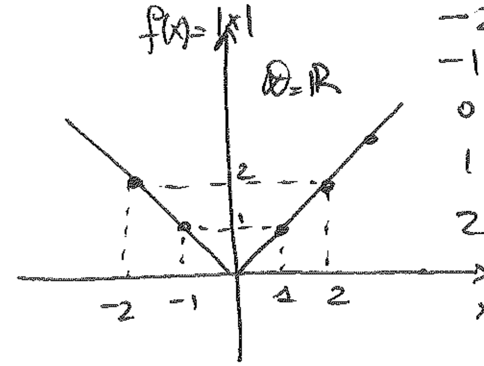
x	$\sqrt{x}$
0	0
1	1
2	1.41
3	1.73
4	2
9	3

As well as

$\mathcal{D} = \{x \geq 0\} = [0, +\infty)$



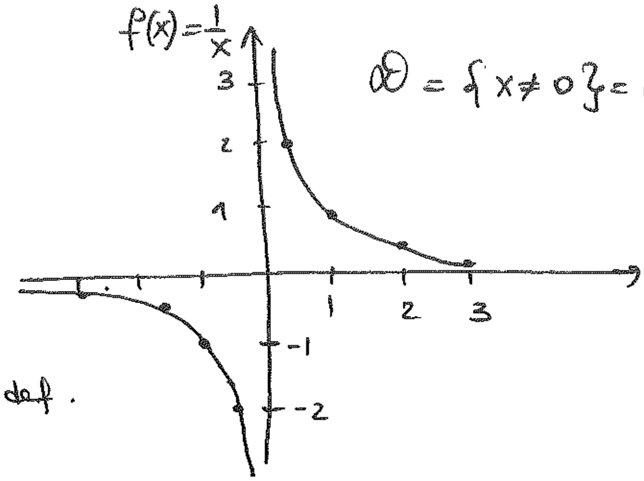
$\mathcal{D} = \mathbb{R}$

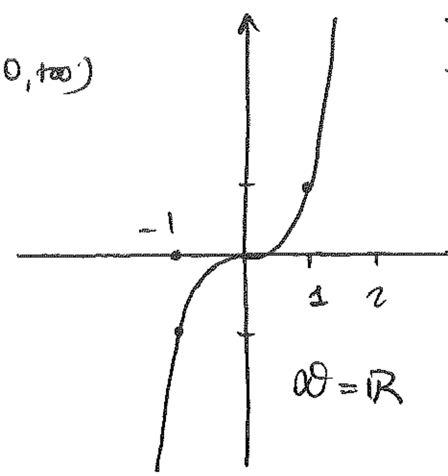


x	$\sqrt{x}$
−2	2
−1	1
0	0
1	1
2	2

−3	−1/3
−2	−1/2
−1	−1
$-\frac{1}{2}$	−2
0	Not def.
$\frac{1}{2}$	2
1	1

$\mathcal{D} = \{x \neq 0\} = (-\infty, 0) \cup (0, +\infty)$





$\mathcal{D} = \mathbb{R}$

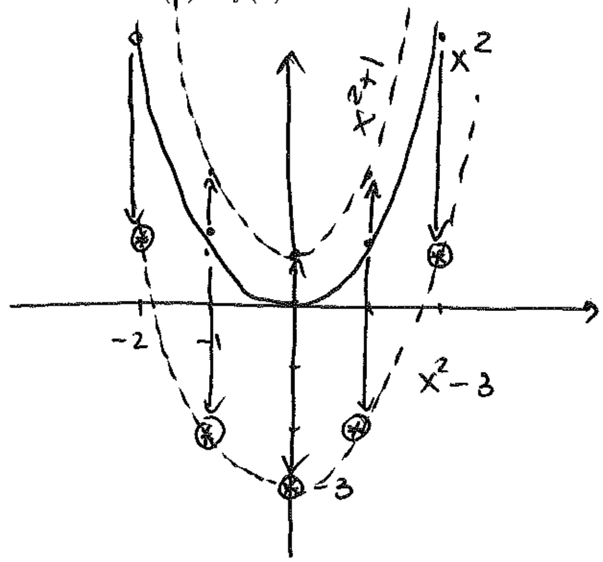
$f(x)$	x^3
−2	−8
−1	−1
0	0
1	1
2	8

Simple operations on functions, and their geometric interpretation.

Once the graphs of basic functions are known, it is quite easy to find out what the graph of various other related functions may be. Let us go through a list of simple examples.

- *Adding or subtracting a number to a function.*
  Let's consider for instance the simple basic function $f(x) = x^2$, and then construct two other functions $g(x) = f(x) + 1 = x^2 + 1$ and $h(x) = f(x) - 3 = x^2 - 3$.

x	x^2	x^2+1	x^2-3
−2	4	5	1
−1	1	2	−2
0	0	1	−3
1	1	2	−2
2	4	5	1



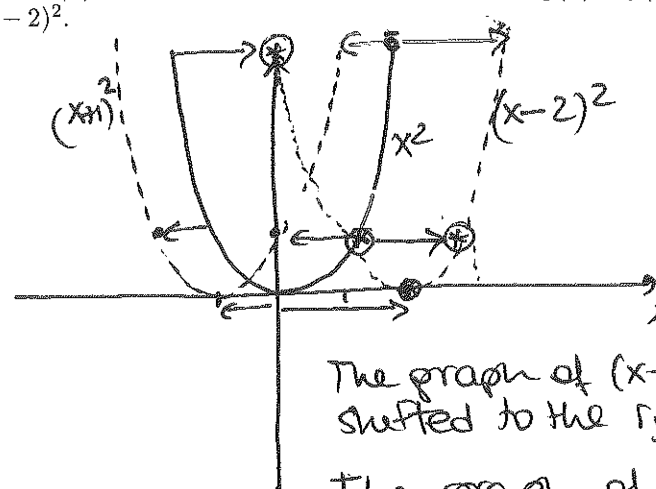
x^2+1 has a graph shifted up by one compared with x^2

x^2-3 has a graph shifted down by 3 one compared with x^2.

This can easily be generalized as : The graph of the function $g(x) = f(x) + a$ can be obtained from the graph of $f(x)$ by shifting it up by an amount a if $a > 0$, and down by an amount $|a|$ if $a < 0$.

- *Applying a function to $x + a$ or $x - a$.*
  Let's consider again the basic function $f(x) = x^2$, and now construct the functions $g(x) = f(x+1) = (x+1)^2$ and $h(x) = f(x-2) = (x-2)^2$.

x	x^2	$(x+1)^2$	$(x-2)^2$
-2	4	1	16
-1	1	0	9
0	0	1	4
1	1	4	1
2	4	9	0



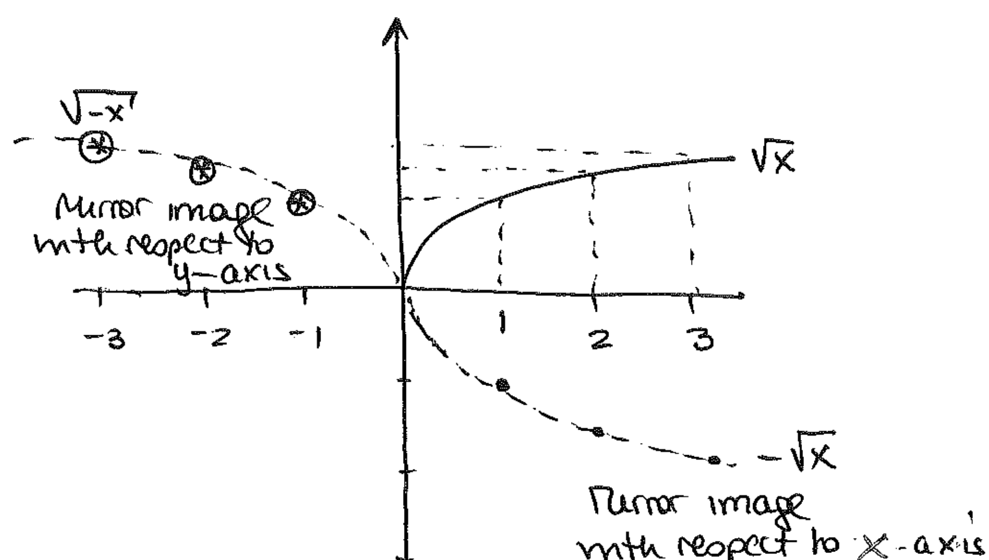
The graph of $(x-2)^2$ is shifted to the <u>right</u> by 2

The graph of $(x+1)^2$ is shifted to the <u>left</u> by 2

Again, this can easily be generalized as: The graph of $g(x) = f(x+a)$ is obtained from that of $f(x)$ by shifting it to the left by a if $a > 0$ and to the right by $|a|$ if $a < 0$.

- *The functions $-f(x)$ and $f(-x)$.*
  This time, we will consider the basic function $f(x) = \sqrt{x}$, and construct $g(x) = -f(x) = -\sqrt{x}$, as well as $h(x) = f(-x) = \sqrt{-x}$.

x	$\sqrt{x}$	$-\sqrt{x}$	$\sqrt{-x}$
-3			1.72
-2			1.41
-1			1
0	0	0	0
1	1	-1	
2	1.41	-1.41	
3	1.72	-1.72	

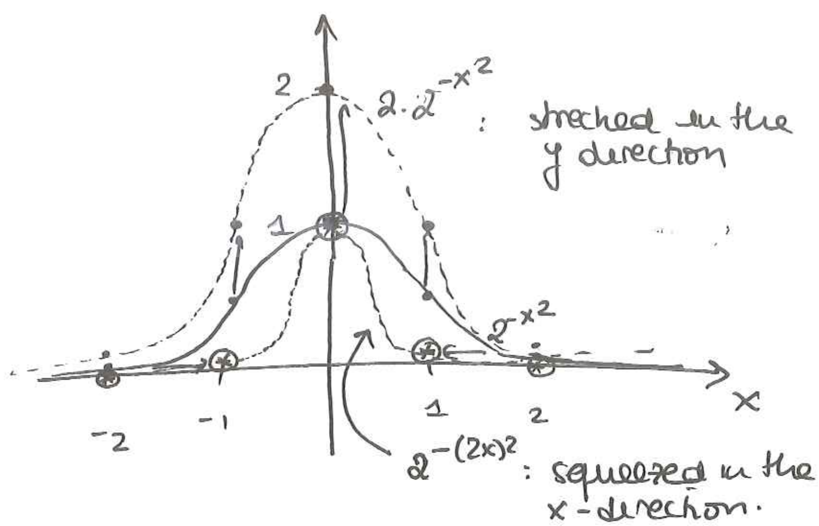


This can easily be generalized as :

- The graph of the function $g(x) = -f(x)$ is the mirror image of the graph of $f(x)$ across the x-axis
- The graph of the function $g(x) = f(-x)$ is the mirror image of the graph of $f(x)$ across the y axis

- *Scaling the dependent or independent variables.*
  To illustrate this case, let's consider another function that we will study in more detail later in this course: $f(x) = 2^{-x^2}$. We then use this function to construct both $g(x) = 2f(x) = 2 \times 2^{-x^2}$, as well as $h(x) = f(2x) = 2^{-(2x)^2} = 2^{-4x^2}$.

x	2^{-x^2}	$2 \cdot 2^{-x^2}$	2^{-4x^2}
-2	$\frac{1}{16}$	$\frac{1}{8}$	$2^{-16} \simeq 0$
-1	$\frac{1}{2}$	1	$\frac{1}{16}$
0	1	2	1
1	$\frac{1}{2}$	1	$\frac{1}{16}$
2	$\frac{1}{16}$	$\frac{1}{8}$	$2^{-16} \simeq 0$

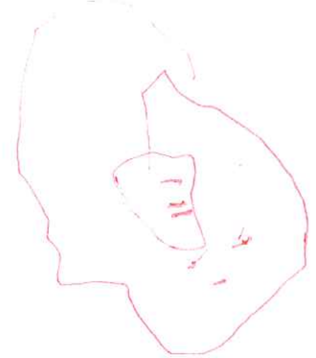


This can easily be generalized as :

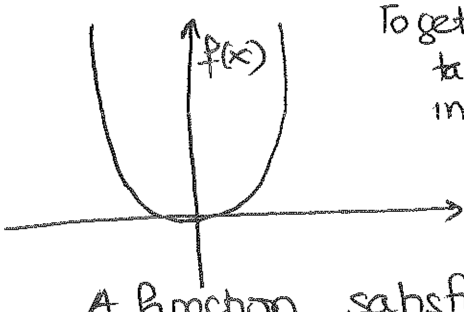
The graph of the function $g(x) = a f(x)$ can be obtained from that of $f(x)$ by stretching ($a>1$) or squeezing ($a<1$) it in the y-direction.
The graph of $g(x) = f(ax)$ can be obtained from that of $f(x)$ by stretching ($a<1$) or squeezing ($a>1$) in the x-direction.

Even and odd functions

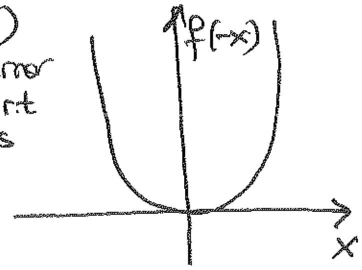
It is worth noting that some graphs have interesting symmetries. For instance the graphs of $f(x) = x^2$ and $f(x) = |x|$ are mirror-symmetric across the y-axis, while the graphs of $f(x) = x^3$ and $f(x) = 1/x$ are point symmetric across the origin. Not all graphs have these symmetries, however. For instance $f(x) = \sqrt{x}$ does not have any special symmetry. Let us now see how the symmetry of the graph relates to the mathematical formula for the function.



Even functions. Consider a function $f(x)$ whose graph is mirror-symmetric across the y-axis. What is the graph of $f(-x)$?



To get $f(-x)$ take mirror image w.r.t y axis $\longrightarrow$

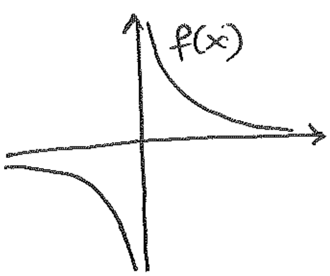


Same graph!

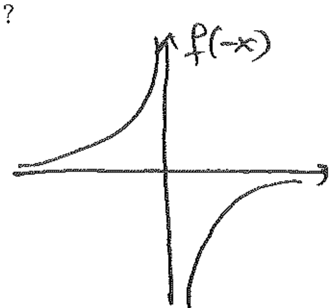
$\Rightarrow$ So this proves $f(x) = f(-x)$

A function satisfying $f(x) = f(-x)$ is an <u>even</u> function as its graph is symmetric about the y-axis

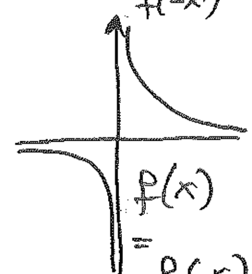
Odd functions. Consider a function $f(x)$ whose graph is point-symmetric across the origin. What is the graph of $f(-x)$? What is the graph of $-f(-x)$?



to get $f(-x)$ take mirror image about y-axis $\longrightarrow$



to get $-f(-x)$ take mirror image about x-axis $\longrightarrow$



$f(x) = -f(-x)$

A function satisfying $f(-x) = -f(x)$ is an <u>odd</u> function. Its graph is symmetric about the origin

EXAMPLES: Are these functions odd or even or neither?

Idea: always calculate $f(-x)$
- if $f(-x) = f(x)$ then its even
- if $f(-x) = -f(x)$ then it's odd
- otherwise it's neither

- $f(x) = 3x$ $\qquad f(-x) = 3(-x) = -3x = -f(x) \Rightarrow$ odd
- $f(x) = -x + 1$ $\qquad f(-x) = -(-x) + 1 = x + 1 \Rightarrow$ neither
- $f(x) = x^4$ $\qquad f(-x) = (-x)^4 = (-1)^4 (x)^4 = x^4 = f(x) \Rightarrow$ even
- $f(x) = -x^5$ $\qquad f(-x) = -(-x)^5 = -(-1)^5 x^5 = x^5 = -f(x) \Rightarrow$ odd
- $f(x) = \frac{2}{x}$ $\qquad f(-x) = \dfrac{2}{-x} = -\dfrac{2}{x} = -f(x) \Rightarrow$ odd
- $f(x) = -\frac{4}{x^2}$ $\qquad f(-x) = -\dfrac{4}{(-x)^2} = -\dfrac{4}{x^2} = +f(x) \Rightarrow$ even.