5.3 The natural exponential and the natural logarithm

As we have seen in the previous lecture, it is quite common to end up having to evaluate logarithms in all sorts of different bases. But if you look carefully on your calculator, you will only see two types of logs: the function **log**, which is actually the log in base 10, and the function **ln** which is another logarithm called the *natural logarithm*, that we will study now. As we will see in this lecture, it is possible to express the logarithm in base a in terms of the natural logarithm, and this is how quantities such as $\log_{1.01}(2)$ or $\log_2(3)$ can be calculated in practice.

Similarly, in scientific studies, it is very rare to use exponentials in many different bases. Instead, scientists tend to express any exponential function they come across in a new base called the *natural exponential.* We will now define both the natural exponential and the natural logarithm.

5.3.1 Definitions

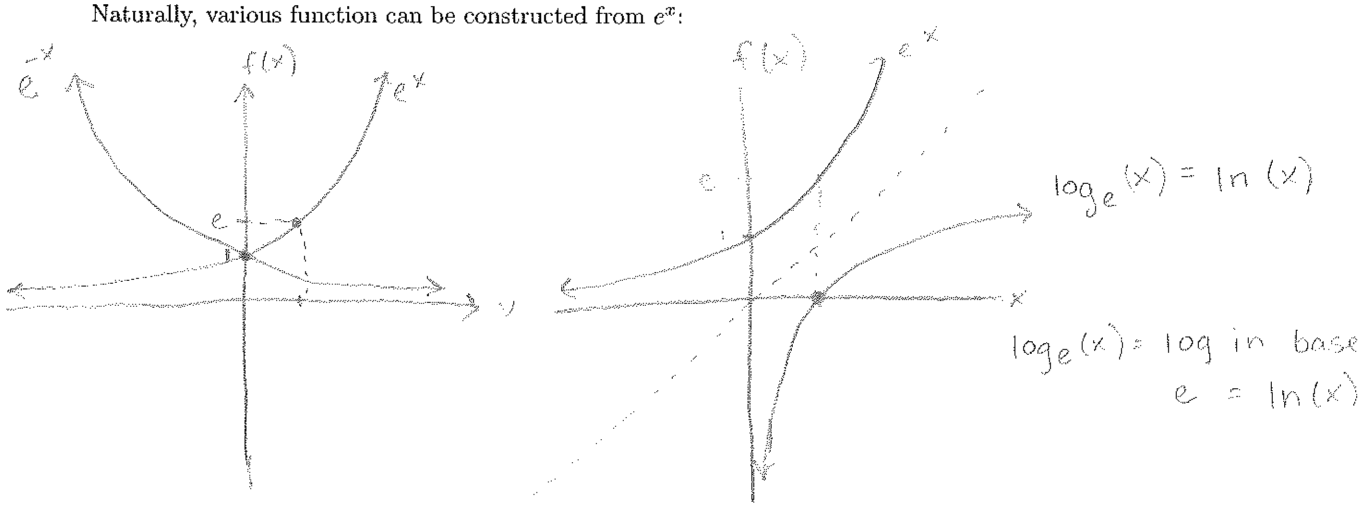
DEFINITION: The natural exponential is the exponential in base e, written as $f(x) = e^x$

(on calculators, written as exp)

The number e is a real number, with value approximately equal to: 2.71...

The reason why this peculiar base is important in mathematics will be explored in more detail in Calculus, but just for a peek, note that the derivative of the function $f(x) = e^x$ (if you know what a derivative is) is *also* e^x, and this is the only function which is also its own derivative – and this is why it is so special.

Naturally, various function can be constructed from e^x:



DEFINITION: The natural logarithm is the inverse of the natural exponential, and is written as

$f(x) = \ln(x)$:

if $y = e^x$ then $x = \ln y$

PROPERTIES OF THE NATURAL LOGARITHM AND EXPONENTIAL: since these two functions are inverse of each other...

- $e^{\ln x} = x$
- $\ln(e^x) = x$ $\quad$ very important formulas.
- $\ln(1) = 0$ $\qquad \ln(e) = 1$

5.3.2 Changing from base a to the natural exponential

Textbook Section 4.5

To change base from a base a exponential to the natural exponential (and vice versa, if needed):

Formula $\quad a^b = e^{b \cdot \ln a}$ $\quad$ for any numbers a & b

(take base and put in exponent as $\ln a$)
b stays where it is.

The reason why this works is simple:

Two methods

① $e^{b \ln a} = e^{\ln a \cdot b} = (e^{\ln a})^b = a^b$ ✓

or ② $a^b = e^{b \ln a} \Rightarrow \ln(a^b) = \ln(e^{b \ln a})$

$b \ln(a) = b \ln(a)$ ← obviously true so original is true!

EXAMPLES:

- $2^x = e^{x \ln 2}$
- $(\frac{1}{4})^x = e^{x \ln(\frac{1}{4})} = e^{x \ln(4^{-1})} = e^{-x \ln(4)}$
- $2 \times 3^{-x} = 2 \cdot e^{-x \ln 3}$
- $4 \times 2^{-2x} = 4 \cdot e^{-2x \ln(2)}$

5.3.3 The growth and decay rates

Using the change of base formulas, we can ultimately express any growing or decaying exponential into one that is in base e:

If $\quad f(x) = ba^x \quad$ then $\quad f(x) = b \cdot e^{x \ln a}$

$f(x) = ba^{-x} \quad$ then $\quad f(x) = b \cdot e^{-x \ln a}$

In other words, our reference formula for exponentials can be rewritten as

$f(x) = be^{rx}$ or $f(x) = be^{-rx}$ where $r = \ln a$ in previous formula

where the plus or minus signs are used depending on whether the exponential is growing or decaying.

The constant r is called "the growth rate" of the exponential if it's a growing exponential or "the decay rate" if it's a decaying exponential

5.3.4 The doubling and halving constants.

Consider a growing exponential $f(x) = be^{rx}$. where $r > 0$

The doubling constant is the number that one must add to x in order to double $f(x)$: it's the number c such that

$$f(x+c) = be^{r(x+c)} = 2f(x) = 2be^{rx}$$

Note that if x is a time, the doubling constant is called the doubling time. If x is a length, then the doubling constant is called the doubling length. Similarly, if we consider a decaying exponential $f(x) = be^{-rx}$ then

The halving constant is the number one must add to x in order to halve $f(x)$: it's the number c such that

$$f(x+c) = be^{-r(x+c)} = \frac{1}{2}f(x) = \frac{1}{2}be^{-rx}$$

The doubling (or halving) constant only depends on the growth (or decay) rate r. To see this, let us solve for the doubling constant: and the halving constant

- Growing exponential: $\cancel{b}e^{r(x+c)} = 2\cancel{b}e^{rx} \Rightarrow e^{rx}e^{rc} = 2\cancel{e^{rx}}$

$$\Rightarrow e^{rc} = 2 \Rightarrow \ln(e^{rc}) = \ln(2) \Rightarrow rc = \ln(2)$$

$$\boxed{c = \frac{\ln(2)}{r}}$$

- Decaying exponential: $\cancel{b}e^{-r(x+c)} = \frac{1}{2}\cancel{b}e^{-rx}$

$$\Rightarrow \cancel{e^{-rx}}e^{-rc} = \frac{1}{2}\cancel{e^{-rx}}$$

$$\Rightarrow e^{-rc} = \frac{1}{2}$$

$$\Rightarrow \ln(e^{-rc}) = \ln\left(\frac{1}{2}\right)$$

$$\Rightarrow -rc = \ln\left(\frac{1}{2}\right) = -\ln(2) \Rightarrow \boxed{c = \frac{\ln(2)}{r}}$$

We see that the same formula applies for both growing and decaying exponentials! In addition, the formula is also independent of x, which means that:

- The time/distance it takes for a growing exponential to double is constant and equal to $\ln(2)/r$!
- The time/distance it takes for a decaying exponential to halve is ~~constanfant~~ constant and equal to $\ln(2)/r$!

EXAMPLE: What is the doubling time for a bank account with an interest rate of 3%?

$m(n) = I_{\text{initial}} \cdot (1.03)^n$ (see previous lecture)

so $= I_{\text{initial}}\, e^{n \ln(1.03)}$ so here $r = \ln(1.03)$

The time it takes to double the amount in the account is $\dfrac{\ln(2)}{r} = \dfrac{\ln(2)}{\ln(1.03)} = 23.449$ years (as we found before but using $\log_{1.03}(2)$)!

5.3.5 Changing from base a to the natural logarithm

Just as we had a rule to change an exponential in base a to the natural exponential, there is also a rule to change a logarithm in base a to the natural logarithm:

$$\log_a(x) = \frac{\ln(x)}{\ln(a)}$$

The reason why this works is simple too:

Exponentiate both sides

$$a^{\log_a(x)} = a^{\frac{\ln(x)}{\ln(a)}} = e^{\frac{\ln(x)}{\cancel{\ln(a)}}\cancel{\ln(a)}} = e^{\ln x} = x$$

$x = x \leftarrow$ trivial so original is true

EXAMPLES:

- $\log_2(x) = \dfrac{\ln x}{\ln(2)}$
- $\log_{\frac{1}{4}} x = \dfrac{\ln x}{\ln(1/4)} = \dfrac{\ln(x)}{-\ln(4)} = -\dfrac{\ln x}{\ln(4)}$

NOTE: This formula, when applied to a, yields the obvious relationship

$$\log_a(a) = \frac{\ln(a)}{\ln(a)} = 1$$

$\hookrightarrow 1 = 1$ ✓

If you are not sure of your change-of-base formula, this is a good way of double-checking that the formula you remember is the correct one.

This change of base is particularly useful because most calculators only provide $\ln(x)$ and not $\log_a(x)$. So, whenever you have to calculate $\log_a(x)$, you can use the formula to evaluate it using a normal calculator.

EXAMPLE:

- How did I evaluate $\log_{1+r}(2)$?

$$\log_{1+r}(2) = \frac{\ln(2)}{\ln(1+r)}$$

- What is $\log_2(3)$?

$$\log_2(3) = \frac{\ln 3}{\ln 2}$$

- Solve the equation $2^x = 6$ and express the result as a natural logarithm.

Solution 1 $2^x = 6 \rightarrow \log_2(2^x) = \log_2(6) \rightarrow x = \log_2(6)$

Solution 2: $2^x = 6 \rightarrow \ln(2^x) = \ln 6 \rightarrow x\ln 2 = \ln 6$

$\rightarrow x = \frac{\ln b}{\ln 2}$ same

- Show that for any a and b, the following is true: $\log_a(b)\log_b(a) = 1$.

$$\log_a(b) \cdot \log_b(a) = \frac{\cancel{\ln b}}{\cancel{\ln a}} \cdot \frac{\cancel{\ln a}}{\cancel{\ln b}} = 1$$

5.3.6 Case study: *Exponential population growth*

In ecology, it is quite common to attempt to model the evolution of the size of the population of a given species under various model assumptions. The simplest such model is called the Exponential growth model (for reasons that will become clear shortly) and the premise of this model is to assume that the population of the species studied multiplies by a given factor over a known reproduction time, without any deaths or loss of reproductivity.

In this case study, we will look at the evolution of a population of rabbits in the wild. A pair of rabbits have on average 6 offsprings every 3 months. Based on this, what is the function that describes the evolution of a population of rabbits with initially one breeding pair, as a function of the number of months elapsed?

2 bunnies $\xrightarrow{\text{3 months}}$ 8 bunnies $\xrightarrow{\text{3 months}}$ 32 bunnies $\longrightarrow$

$\rightarrow$ Every 3 months, the population multiples by 4.

$n = 0 : 2$

$n = 3 : 8 \quad \rightarrow \quad N(n) = 2 \cdot (4)^{\frac{n}{3}}$ ✓

$n = 6 : 32$

$n = 9 : 128$

number of rabbits

number of months

Let's now recast this function as a natural exponential: Use: $a^b = e^{b \cdot \ln a}$

$$N(n) = 2 \cdot e^{\frac{n}{3}\ln 4} = 2e^{0.462n}$$

We now see that the growth rate of the rabbit population is

Growth rate is $r = \frac{\ln 4}{3} \simeq 0.462$ → These growth rates are intrinsic to each species & can be compared.

We can also answer further questions such as how long does it take for the rabbit population to double?

→ Doubling time is related to the growth rate r as

$$T_{double} = \frac{\ln 2}{r} = \frac{\ln 2}{\frac{\ln 4}{3}} = 3 \cdot \frac{\ln 2}{\ln 4} = 1.5 \text{ months}$$

$(\ln 4 = \ln(2^2) = 2\ln 2)$

How long does it take for this rabbit population to exceed the human population of the Earth (estimated at 7 billion)?

→ Solve $7 \times 10^9 = 2 \cdot (4)^{\frac{n}{3}}$

→ $\frac{7 \times 10^9}{2} = 4^{\frac{n}{3}} = 3.5\ 10^9$

→ take ln of both sides $\ln\left(4^{\frac{n}{3}}\right) = \ln(3.5\ 10^9)$

→ $\frac{n}{3}\ln 4 = \ln(3.5\ 10^9)$ → $n = 3\,\frac{\ln(3.5\ 10^9)}{\ln 4}$

$\cong 47.55$ months (4 years).

What could be wrong about this rabbit population model?

- It forgets about predators
- It forgets about disease/deaths
- It ignores the environmental limitations (i.e how much food is needed to support such a large population).