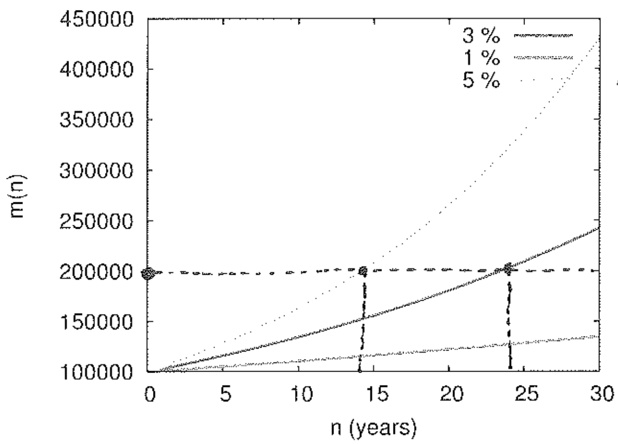
5.2 General logarithmic functions

Textbook Section 4.4

5.2.1 Case Study 1: *Compound interests*

Let's now go back to our first Case study of the previous lecture, and ask a simple question: how much time will it take, for a given interest rate, to double our initial \$100,000 investment? The following graph shows the function $m(n)$. We see that:



To double the initial investment we need to wait until we have \$200,000 in the account.

- about 14 yrs at 5% rate
- about 24 yrs at 3% rate
- much longer than 30 yrs at 1% rate.

However, this graphical method is not very precise, and with the available graph, is also not working for the case of the 1% interest rates. We could try to solve the problem mathematically, in which case, what equation would we need to solve to determine how long it takes to double the initial investment?

General formula for money in bank as function of years n and rate r was

$$m(n) = 100{,}000\,(1+r)^n$$

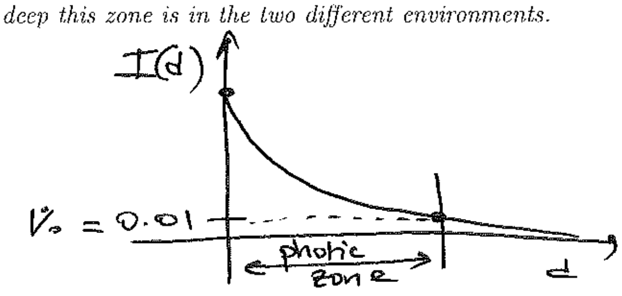
$\Rightarrow$ Solve $m(n) = 200{,}000$ for n: $\Rightarrow \boxed{200{,}000 = 100{,}000\,(1+r)^n}$

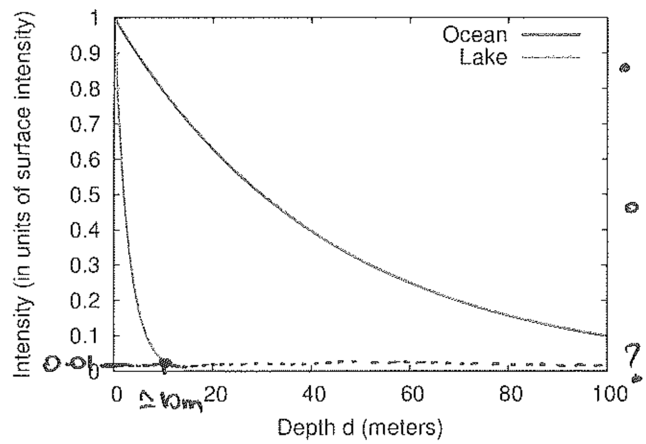
These types of equations are called exponential equations – note how the variable we are trying to solve for is in the exponent. But they are also equations that have been obtained by asking a question of the kind " for a given value of y, which value of x did it come from?". In other words, this equation could be solved if we knew what the inverse of an exponential function was.

A similar equation can be obtained from our other case study as well.

5.2.2 Case Study 2: *The photic zone in the Ocean*

In our previous lecture, we studied how light is attenuated as a function of depth below the surface of the water in both the clear open ocean, and in a rather murky lake. The bottom of the photic zone is usually defined to be where the light intensity drops to about 1% of its surface value. We can therefore wonder how deep this zone is in the two different environments.



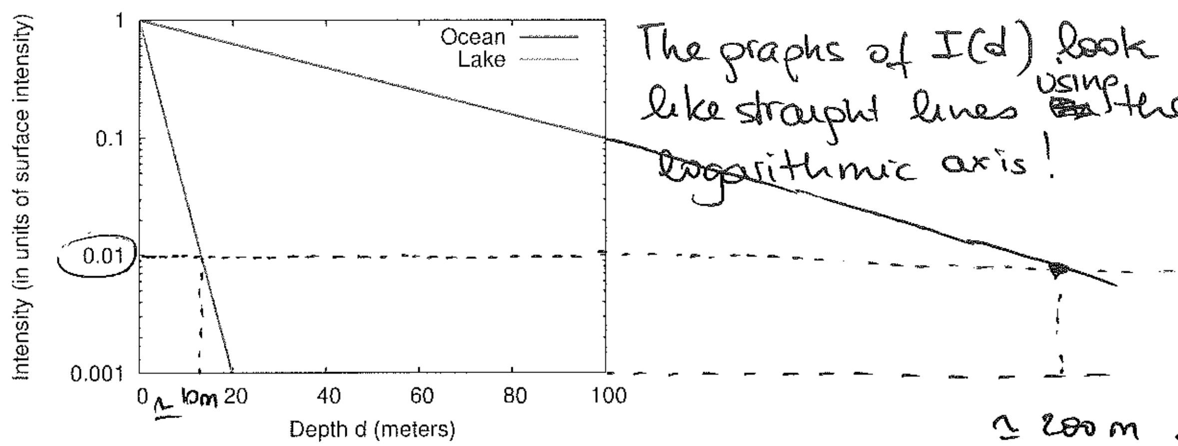


• Bottom of photic zone is about at 10 m depth for lakes.

• For ocean case, we can't read this information from the graph.

Here, the graphical method is not very precise at all, because we can hardly see on this graph where the point 0.01 is on the y− axis. But we can use a trick that is similar to the one we used in the Case Study about the Rank-Size distribution of cities : logarithmic axes. To be precise, let's use a logarithmic axis for the y-axis only, and re-plot the data. We see that something remarkable happens (which will be explained next week):

The graphs of $I(d)$ look like straight lines using the logarithmic axis!



Thanks to this, we can get a better estimate the depth of the bottom of the photic zone for the lakes and even for the ocean (with a bit of extrapolation):

Lakes: photic zone ~ 10m deep; Ocean photic zone ≃ 200m deep.

Still, this is not very precise. Going back to the mathematical expression for the intensity of light, what mathematical equation would we have to solve in order to find the depth at which the light intensity drops to 1 percent of the surface intensity?

Lake: $I(d) = \left(\frac{1}{2}\right)^{d/2}$

Ocean: $I(d) = \left(\frac{1}{2}\right)^{d/30}$

Intensity is 0.01 when

$\Rightarrow 0.01 = I(d)$

Lakes: $0.01 = \left(\frac{1}{2}\right)^{d/2}$

Ocean: $0.01 = \left(\frac{1}{2}\right)^{d/30}$

This kind of equation is again one in which the variable is in the exponent, and again has been obtained by asking the question "For a given value of y, which x did it come from?". In other words, this is exactly the same situation as before, which requires knowledge of the inverse of an exponential function. So let's now find out what they are!

5.2.3 Definition and graph

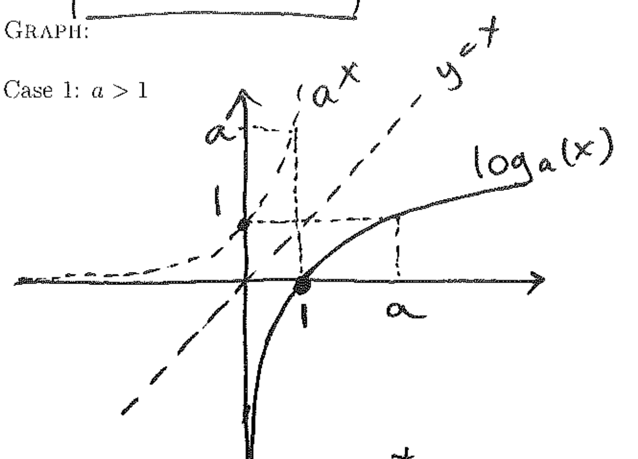
DEFINITION: The logarithm in base a is defined to be the inverse of the exponential in base a

So: if $\boxed{y = a^x \leftrightarrow x = \log_a(y)}$

$\boxed{\text{Note: } a > 0}$

GRAPH:

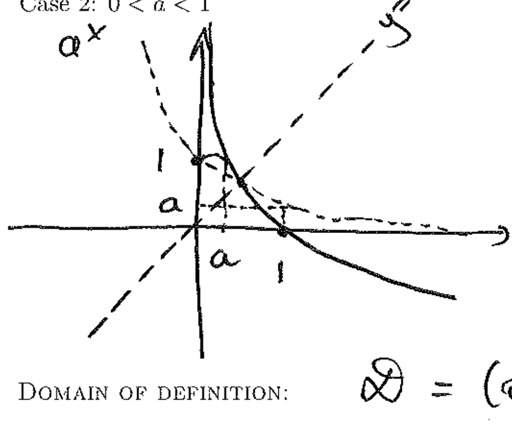
Case 1: $a > 1$



- $\log_a(1) = 0$
- $\log_a(a) = 1$
- $\log_a(x)$ is only define if $x > 0$

$\mathcal{D} = (0, +\infty)$

Case 2: $0 < a < 1$



- $\log_a(1) = 0$
- $\log_a(a) = 1$

(rarely ever used)

DOMAIN OF DEFINITION: $\mathcal{D} = (0, +\infty)$

UNIVERSAL PROPERTY OF LOGARITHMS:

- $\log_a(1) = 0$
- $\log_a(a) = 1$

5.2.4 Examples of logarithms in common bases

THE FUNCTION $f(x) = \log_2(x)$ (LOGARITHM IN BASE 2)

x	$y = 2^x$
-3	$1/8$
-2	$1/4$
-1	$1/2$
0	1
1	2
2	4
3	8

switch $\longrightarrow$ columns

x	$\log_2(x)$
$1/8$	-3
$1/4$	-2
$1/2$	-1
1	0
2	1
4	2
8	3

Finding the $\log_2(x)$ is equivalent to asking the question "For which value of p is $2^p = x$?"

THE FUNCTION $f(x) = \log_{10}(x)$ (LOGARITHM IN BASE 10)

x	10^x
-3	0.001
-2	0.01
-1	0.1
0	1
1	10
2	100
3	1000

switch $\longrightarrow$ columns

x	$\log_{10}(x)$
0.001	-3
0.01	-2
0.1	-1
1	0
10	1
100	2
1000	3

Finding $\log_{10}(x)$ is equivalent to asking the question "For which value of p is $10^p = x$?"

EXAMPLES:

- $\log_{10}(1000) = 3$
- $\log_{10}(0.01) = -2$
- $\log_2(0.25) = -2$
- $\log_2(8) = 3$

$\log_2(128) = 7$

$\log_{10}(1{,}000{,}000{,}000) = 9$

$\log_{10}(10^9) = 9$

$\log_5(125) = 3$

$$f(f^{-1}(x)) = x$$

$$y = f(x) \qquad x = f^{-1}(y) \qquad f^{-1}(f(x)) = x$$

5.2.5 The inverse relationships

Since the logarithm in base a is the inverse of the exponential in base a, we have the two fundamental relationships

- $\log_a(a^x) = x$
- $a^{\log_a(x)} = x$

These relationships can be used to solve exponential equations, such as the following examples:

EXAMPLES:

- Solve $3^x = 2$ $\longrightarrow$ apply $\log_3$ to both sides

$$\log_3(3^x) = \log_3(2) \implies x = \log_3(2)$$

- Solve $4^{-x} = 2$ $\longrightarrow$ apply $\log_4$ to both sides

$$\log_4(4^{-x}) = \log_4(2) \rightarrow -x = \log_4(2) = \frac{1}{2}$$

$$\rightarrow x = -\frac{1}{2}$$

- Solve $10^{-x} = -2$

$\longrightarrow$ NO SOLUTION!

$$\log_{10}(10^{-x}) = \log_{10}(-2) \rightarrow$$ does not exist (logs of negative numbers don't exist)

These properties can also be used to solve logarithmic equations, or to simplify expressions with exponentials and logs...

EXAMPLES:

- $\log_2(2^x) = x$

$$\log_2\left(\left(\tfrac{1}{2}\right)^x\right) = \log_2(2^{-x}) = -x$$

- $\log_5(5\sqrt{5}) = \log_5(5 \cdot 5^{\frac{1}{2}}) = \log_5(5^{1+\frac{1}{2}}) = \log_5(5^{\frac{3}{2}}) = \frac{3}{2}$
- $\log_{10}(100^x) = \log_{10}(10^{2x}) = 2x$
- $3^{\log_3(2)} = 2$
- $10^{\log_{100}(2x)} = (100^{\frac{1}{2}})^{\log_{100}(2x)} = \left(100^{\log_{100}(2x)}\right)^{\frac{1}{2}} = (2x)^{1/2}$

$$(a^x)^y = a^{xy} = a^{yx} = (a^y)^x$$

- Solve $\log_2(x) = 3$ → exponentiate on both sides

$$2^{\log_2(x)} = 2^3 \rightarrow x = 8$$

- Solve $2\log_{10}(x) = -4$

$$\log_{10}(x) = -\frac{4}{2} \rightarrow 10^{\log_{10}(x)} = 10^{-2} \rightarrow x = 0.01$$

- Solve $2\log_{10}(-x) = -4$

5.2.6 Case Studies 1 and 2

Let us now go back to the two case studies and solve the equations we arrived at to answer the questions that were posed.

How long does it take to double the initial investment? *To answer this question we solve the equations:*

For 5% investment: Solve $200{,}000 = 100{,}000(1.05)^n$ for n

$$\rightarrow \frac{200000}{100000} = 1.05^n \rightarrow 2 = 1.05^n \rightarrow \log_{1.05}(2) = n$$

$$\rightarrow n = \log_{1.05}(2) = 14.2 \text{ years.} \qquad \log_{1.05}(1.05^n)$$

For 3% investment: $200{,}000 = 100{,}000(1.03)^n$ for $n \rightarrow n = \log_{1.03}(2)$ $= 23.45$ yrs

For 1% investment: $n = \log_{1.01}(2) = 69.66$ yrs.

In fact, we can even solve a much more interesting question, namely: how long does it take to double any initial investment as a function of the interest rate r?

- Given an interest rate r, and an initial investment of I_{init} dollars; the amount of money in the account would be $I_{init} \cdot (1+r)^n$
- How long does it take to double the investment?

Solve $\underbrace{2 \cancel{I_{init}}}_{\text{doubled investment}} = \cancel{I_{init}}(1+r)^n$ for n

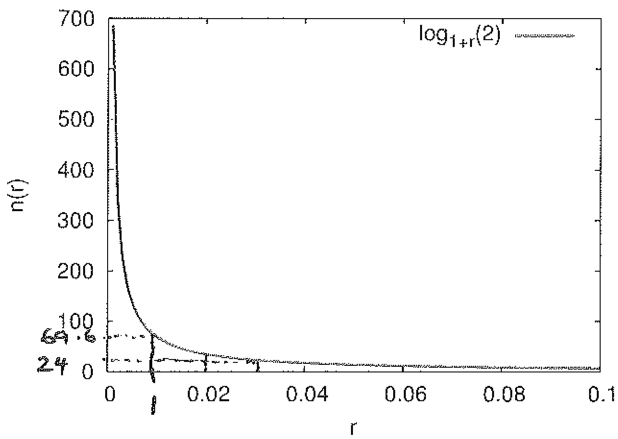
$$\hookrightarrow 2 = (1+r)^n \rightarrow \boxed{n = \log_{1+r}(2)}$$

$$\log_{1+r}(2) = \underbrace{\log_{1+r}((1+r)^n)}_{n}$$

We see that, very generally,

- The time it takes to double the investment is independent of the investment
- It only depends on the interest rate.

Here is a graph showing the doubling time $n(r)$ as a function of r. You may wonder how this graph was even plotted, given that most graphing calculators do not know what log is in any base. More on this in the next lecture.



→ The time it take to double the investment is extremely sensitive to the interest rate!

How deep is the photic zone of the ocean and of a lake? *To answer this question we solve the equations:*

Lake: $0.01 = \left(\frac{1}{2}\right)^{\frac{d}{2}} = 2^{-\frac{d}{2}}$

→ $\log_2(0.01) = \log_2\left(2^{-\frac{d}{2}}\right) = -\frac{d}{2}$

→ $d = -2\log_2(0.01) = 13.3\,m$

Ocean: $0.01 = \left(\frac{1}{2}\right)^{\frac{d}{30}} = 2^{-\frac{d}{30}}$

$\log_2(0.01) = \log_2\left(2^{-\frac{d}{30}}\right) = -\frac{d}{30} \rightarrow d = -30\log_2(0.01) = 199.\,m$

In fact, we can even solve a much more interesting question, namely: how deep is the photic zone as a function of the decay rate l of the light intensity?

Let's now go back and learn a little more about Logarithmic functions in general.

5.2.7 Properties of the logarithms and examples of use

Textbook Section 4.5

The following rules apply for logarithms.

- $\log_a(xy) = \log_a(x) + \log_a(y)$
- $\log_a(x^b) = b\log_a(x)$
- $\log_a\left(\frac{1}{x}\right) = -\log_a(x)$
- $\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$

To show why these formulas are true, we go back to the definition of the logarithm as an inverse, and use the properties of the exponentials: for instance, to show why the first formula is true write:

exponentiate both sides

$$\log_a(xy) = \log_a(x) + \log_a(y)$$

$$a^{\log_a(xy)} = a^{\log_a(x) + \log_a(y)} = a^{\log_a(x)} \cdot a^{\log_a(y)}$$

$xy = x \cdot y$ ← this is obviously true, so original equation must be true!

Similarly, we can also show why the second formula is true:

exponentiate

$$\log_a(x^b) = b\log_a(x)$$

$$a^{\log_a(x^b)} = a^{b\log_a(x)} = \left[a^{\log_a(x)}\right]^b$$

$x^b = x^b$ ← again this is obviously true, so original is true!

Then, using this, we can now see why the other ones are true as well: 3

For $\log_a\left(\frac{1}{x}\right) = \log_a(x^{-1}) = -\log_a(x)$ ✓

For $\log_a\left(\frac{x}{y}\right) = \log_a(xy^{-1}) = \log_a(x) + \log_a(y^{-1})$

$$= \log_a(x) - \log_a(y)$$

EXAMPLES:

- Combine into one log expression: $\log_2(x^2-1) - \log_2(x+1)$

$$\log_2(x^2-1) - \log_2(x+1) = \log_2\left(\frac{x^2-1}{x+1}\right) = \log_2\left(\frac{(x-1)(x+1)}{x+1}\right)$$
$$= \log_2(x-1)$$

- Simplify $\log_2(8(x-2)^3)$:

$$\log_2(8) + \log_2((x-2)^3)$$
$$\log_2(2^3) + \log_2((x-2)^3) = 3 + 3\log_2(x-2)$$

- Simplify $\log_3(9^{-x})$:

$$\log_3\left(\frac{1}{9^x}\right) = -\log_3(9^x) = -x\log_3(9) = -2x$$

- Simplify $\log_{10}(100^{x+1}) + \log_{10}\left(\frac{1}{5^x}\right)$

$$\log_{10}(100^{x+1}) + \log_{10}\left(\frac{1}{5^x}\right)$$
$$(x+1)\log_{10}(100) - \log_{10}(5^x)$$
$$2(x+1) - x\log_{10}(5)$$

- Write the following expression as a sum or difference of logs: $\log_{10}\left[\frac{x-1}{(x+2)^2}\right]$

$$\log_{10}\left(\frac{x-1}{(x+2)^2}\right) = \log_{10}(x-1) - \log_{10}((x+2)^2)$$
$$= \log_{10}(x-1) - 2\log_{10}(x+2)$$

- Write the following expression as a sum or difference of logs: $\log_3\left[\frac{3^{x+1}(x-3)^2(x+4)}{(x-2)^3 9^x}\right]$

$$\log_3\left[\frac{3^{x+1}(x-3)^2(x+4)}{(x-2)^3\,9^x}\right] = \log_3(3^{x+1}) + \log_3((x-3)^2) + \log_3(x+4)$$
$$- \log_3((x-2)^3) - \log_3(9^x)$$
$$= x+1 + 2\log_3(x-3) + \log_3(x+4)$$
$$- 3\log_3(x-2) - 2x$$