Chapter 5

Exponentials and logarithms

5.1 General Exponential functions

Textbook Chapter 4.3

5.1.1 Case Study 1: *Compound interests*

When opening a savings account, a bank usually offers an interest rate compounded yearly. Suppose for simplicity that the interest rate is 3%. Compounding yearly means that the interests, at the rate of 3% of the total in your account, are calculated and added to your account once a year. This case study focuses on figuring out how money accrues in your account, starting from a total amount of money of \$100,000.

Starting with \$100,000 how much money will be in the account after 1 year? after 2 years? (assume that no money is taken out in between).

year 0: 100,000

year 1: $100{,}000 + \frac{3}{100} \cdot 100{,}000$ (0.03)

$= 100{,}000(1 + 0.03) = 100{,}000 \cdot (1.03)$

$= 103{,}000$

adding 3% is the same as multiplying by 1.03

year 2: $103{,}000 \cdot (1.03) = 106{,}090$

$= 100{,}000 \cdot (1.03) \cdot (1.03)$

$= 100{,}000 \cdot (1.03)^2$

year 3 $\Rightarrow$ $100{,}000 \cdot (1.03)^2 \cdot (1.03)$

$= 100{,}000 \cdot (1.03)^3$

Based on this, how much money will be in the account after n years (assuming no money is ever taken out of the account)?

→ After n years we expect to have

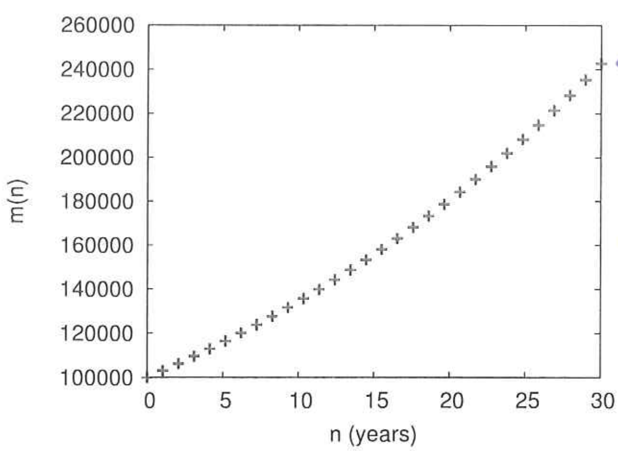
$$m(n) = 100,000 \cdot (1.03)^{n}$$

n is number of years.

Example: After 10 years

$$m(10) = 100,000 \cdot (1.03)^{10} = 134,391 \text{ dollars.}$$

The following graph shows the function m(n). We see that:



- The amount of money in the account increases every year
- The increase is faster than linear (upward curve).

Suppose we now have a more realistic case scenario where the interest rate is 1% instead of 3%. What is the new function that describes the amount of money in your account as a function of year?

→ after year 1 : $100,000 \times 1.01$

" " 2 : $100000 \times 1.01 \times 1.01 = 100,000\,(1.01)^2$

" " 3 : $100,000 \times (1.01)^3$

⋮

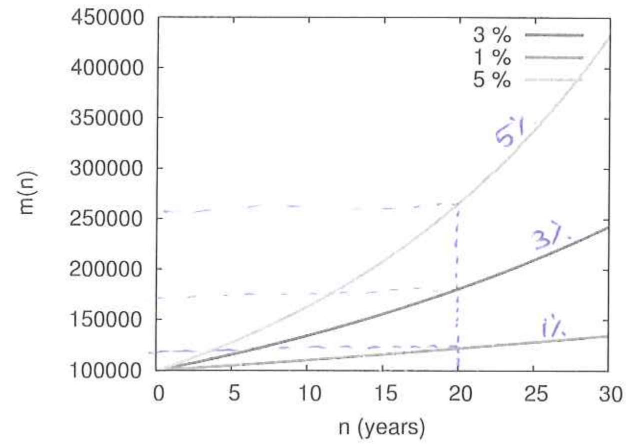
" n years : $100,000 \times (1.01)^n$

Suppose instead we consider an investment account with an estimated growth rate of 5% (and we are lucky enough that the stock market does not crash). What is the new function m(n) now?

→ after n years:

$$100,000 \times (1.05)^n$$

The following graph compares the functions $m(n)$ for the 3 different interest rates. We see that:



After 20 years one would have
- for 1% $\sim 120{,}000 = 100000 \cdot (1.01)^{20}$
- for 3% $\sim 170{,}000 = 100000 \cdot (1.03)^{20}$
- for 5% $\sim 220{,}000 = 100000\,(1.05)^{20}$

→ The interest rate makes a huge difference to how much money is gained after n years.

5.1.2 Case study 2: *The photic zone in the Ocean*

The photic zone is the upper layer of the ocean or a lake where there is enough light to support life, i.e. where phytoplanktons or algae can grow, and become part of the food chain for increasingly large predators. The photic zone is limited to a region near the surface because intensity of sunlight decreases with depth under water, as it gets absorbed by water molecules. This case study works out what fraction of the light intensity at the surface penetrates down to a certain depth, in the clear open ocean, and in a lake.

In the clear open ocean, the light intensity drops by a factor of 2 roughly every 30 meters. In more murky lakes and ponds, the light intensity drops much more rapidly, by a factor of 2 roughly every 2 meters instead. Let us construct a function to model the intensity of light as a function of depth below the surface, in meters, for both cases. For simplicity, let's assume that the intensity of light at the surface is exactly 1. (If you are worried about this assumption, note that we can always do this, by selecting a unit system in which the intensity of light at the surface is the unit intensity).

Let's start with the case of the ocean:

At surface: Intensity $= 1$ depth $= 0$
$I(d=0) = 1$

At 30 m down: Intensity $= \frac{1}{2}$ depth $d = 30$
→ $I(d=30) = \frac{1}{2}$

At 60 m down: Intensity $= \frac{1}{4}$ depth 60
$I(d=60) = \frac{1}{4}$

At 90 m down: Intensity $= \frac{1}{8}$ $I(d=90) = \frac{1}{8}$

d	0	30	60	90	... d
I(d)	1	1/2	1/4	1/8	

$$\boxed{I(d) = \left(\frac{1}{2}\right)^{\frac{d}{30}}} = 2^{-\frac{d}{30}}$$

Based on this construction, let's also construct a function to model the intensity of light as a function of depth below the surface, in meters, for the lakes.

At surface: $d = 0$ $\quad I = 1$

$d = 2$ $\quad I = 1/2$

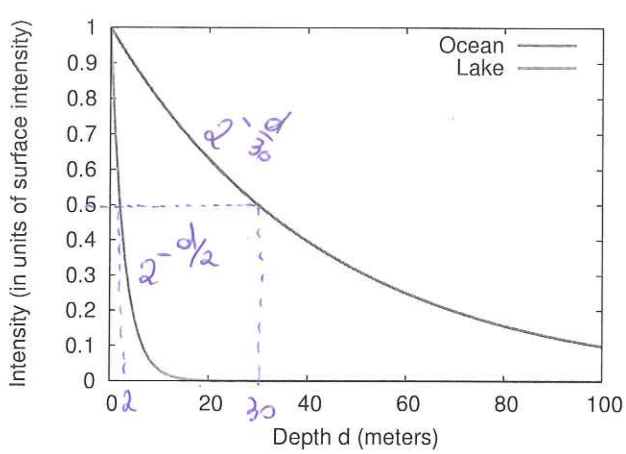
$d = 4$ $\quad I = 1/4$

$d = 6$ $\quad I = 1/8$

$d = 8$ $\quad I = 1/16$

$$\boxed{I(d) = \left(\frac{1}{2}\right)^{\frac{d}{2}} = 2^{-\frac{d}{2}}}$$

The light intensity as a function of depth in both cases in shown in the following graph. We see that:



These functions are exponential functions representing exponential decay of the light intensity (as function of depth).

The functions that we have constructed are all exponential functions. Exponential functions play a crucial role in nearly every aspect of mathematical modeling, in ecology, epidemiology, physics, chemistry, economics (as we just saw) and arise more-or-less whenever a quantity gets multiplied over and over again by the same number as we saw in both case studies. We will now learn about some generic properties of exponentials.

5.1.3 Definition of an exponential functions

DEFINITION: An exponential function is any function of the kind $f(x) = a^x$ or $f(x) = a^{-x}$

The constant a, must be positive, is called the base of the exponential.

NOTE: Do not mix up power and exponential functions!

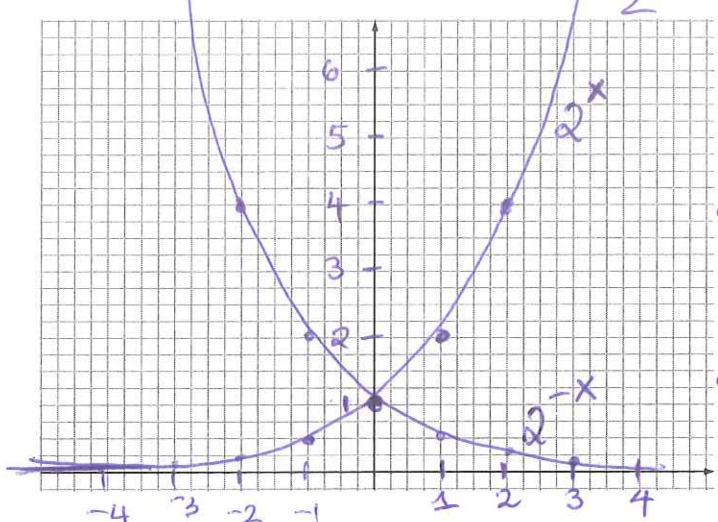
- For power functions: $f(x) = x^a$

- For exponential functions: $f(x) = a^x$ → "x is in exponent" for an exponential function.

5.1.4 Graphs of exponential functions

While we may not be used to thinking of exponents as non-integers, or non-rational numbers, just construct the following tables for the functions $f(x) = 2^x$ and $g(x) = 2^{-x}$:

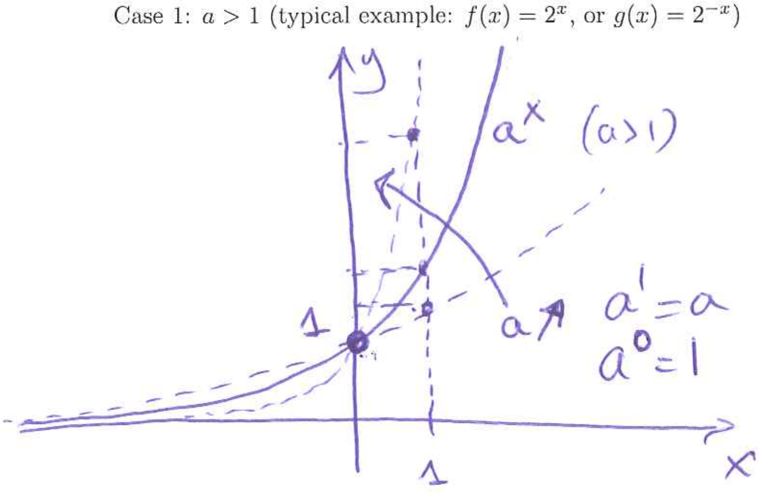
x	-3	-2	-1	0	1	2	3
2^x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
2^{-x}	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

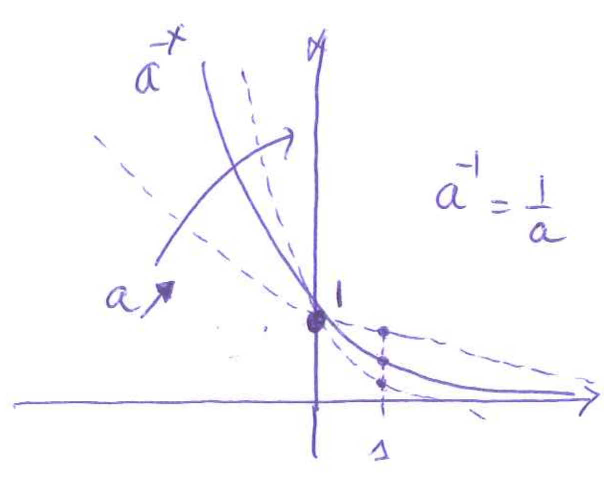


- 2^x increases with x extremely rapidly as $x \to +\infty$
- 2^x has horizontal asymptote at $y=0$ as $x \to -\infty$
- The graph of 2^{-x} is the mirror image of the graph of 2^x across y-axis

More generally, the graph of a basic exponential function $f(x) = a^x$ or $g(x) = a^{-x}$ depends on the value of the base a.

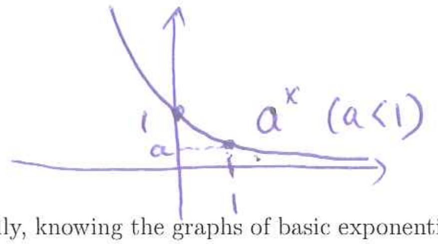
Case 1: $a > 1$ (typical example: $f(x) = 2^x$, or $g(x) = 2^{-x}$)

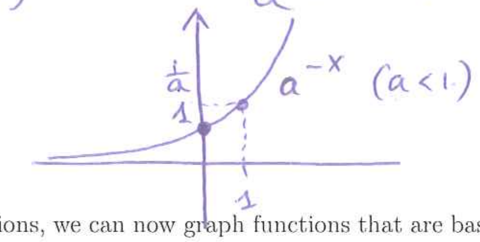




Case 2: $0 < a < 1$ (typical example: $f(x) = 0.5^x$ or $g(x) = 0.5^{-x}$) Recall $0.5^x = \left(\frac{1}{2}\right)^x = 2^{-x}$

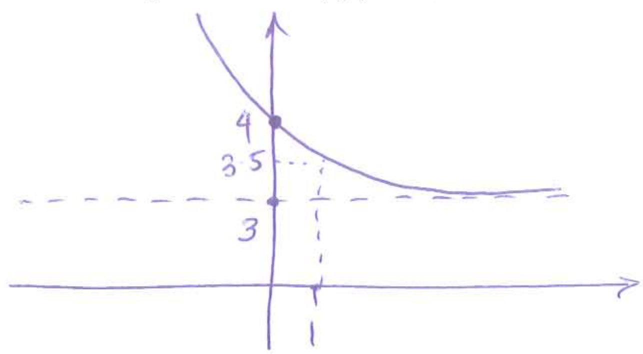
$\rightarrow$ Rewrite a^x as $\left(\frac{1}{a}\right)^{-x}$ where $\frac{1}{a} > 1$ (since $a<1$)





Finally, knowing the graphs of basic exponential functions, we can now graph functions that are based on the latter, through simple geometric transformations. Here are some examples:

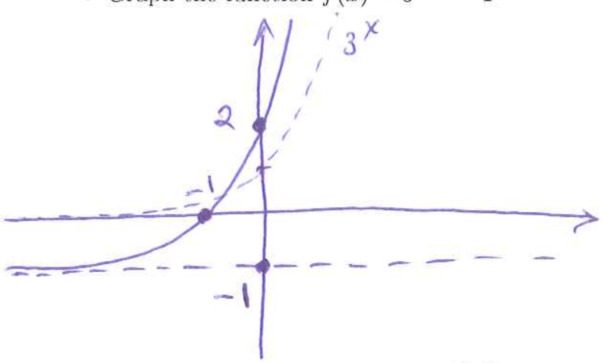
- Graph the function $f(x) = 3 + 2^{-x}$



$f(0) = 3 + 2^0 = 4$

$f(1) = 3 + 2^{-1} = 3.5$

- Graph the function $f(x) = 3^{x+1} - 1$



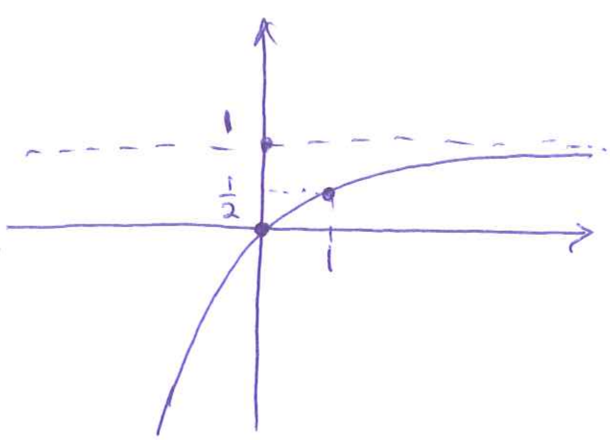
$f(0) = 3^1 - 1 = 2$

$f(1) = 3^2 - 1 = 8$

$f(-1) = 3^0 - 1 = 1 - 1 = 0$

- Graph the function $f(x) = 1 - \left(\frac{1}{2}\right)^x$

$f(x) = 1 - \left(\frac{1}{2}\right)^x = 1 - 2^{-x}$



$f(0) = 1 - 2^0 = 0$

$f(1) = 1 - 2^{-1} = 1 - \frac{1}{2} = \frac{1}{2}$

5.1.5 Properties of exponential functions

MANIPULATION OF EXPONENTIAL FUNCTIONS: The rules for manipulating these functions are the same as the rules for manipulating exponents. Given an exponential function in base a

- $a^0 = 1$
- $a^1 = a$
- $a^{x+y} = a^x a^y$
- $a^{-x} = \frac{1}{a^x}$
- $\frac{a^x}{a^y} = a^{x-y}$
- $(a^x)^y = a^{xy} = a^{yx} = (a^y)^x$

Also, given another exponential function in base b

- $a^x b^x = (ab)^x$
- $\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x} = a^x b^{-x}$

but $a^x b^y$ cannot be simplified

etc...

EXAMPLES:

- Simplify: $f(x) = \frac{3^{x+2}}{9}$

$$f(x) = \frac{3^{x+2}}{9} = \frac{3^x \cdot \cancel{3^2}}{\cancel{9}} = 3^x$$

$$f(x) = \frac{3^{x+2}}{9} = \frac{3^{x+2}}{3^2} = 3^{x+\cancel{2}-\cancel{2}} = 3^x$$

- Simplify $f(x) = \frac{2^{2x}}{4^x}$

$$f(x) = \frac{2^{2x}}{4^x} = \frac{2^{2x}}{(2^2)^x} = \frac{2^{2x}}{2^{2x}} = 1$$

- Simplify $f(x) = 25^x 5^{-x-1}$

$$25^x 5^{-x-1} = (5^2)^x \cdot 5^{-x-1} = 5^{2x} \cdot 5^{-x-1} = 5^{2x-x-1} = 5^{x-1}$$

$$25^x 5^{-x-1} = \frac{25^x 5^{-x}}{5} = \frac{25^x}{5^x \cdot 5} = \left(\frac{25}{5}\right)^x \cdot \frac{1}{5} = 5^x \cdot \frac{1}{5}$$

- Simplify $f(x) = 2^{2x} 3^x$

$$2^{2x} \cdot 3^x = (2^2)^x \cdot 3^x = 4^x \cdot 3^x = 12^x$$