Chapter 4

Powers functions. Composition and Inverse of functions.

4.1 Power functions

We have already encountered some examples of power functions in the previous chapters, in the context of polynomial and rational functions: indeed, functions such as

$f(x) = 3x^2 \qquad f(x) = \frac{7}{x^4} \qquad f(x) = x^{-2} \qquad f(x) = \sqrt{x}$

as power functions. More generally speaking, power functions are defined as follows:

DEFINITION: A power function is any function of the kind $f(x) = ax^b$ where a & b can be any number.

EXAMPLES:

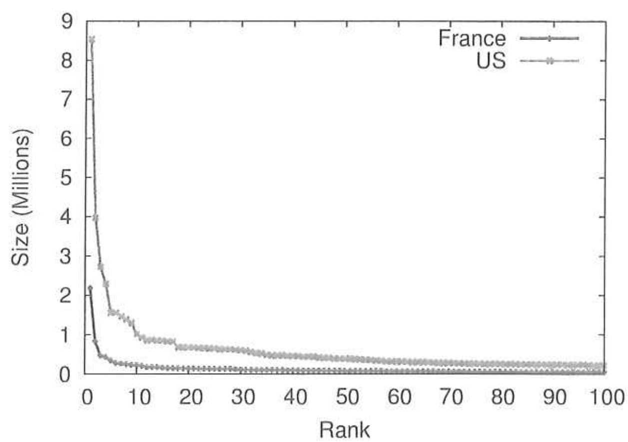
$f(x) = x^{-\frac{1}{8}} \qquad f(x) = 2x^{\frac{8}{5}}$

$f(x) = x^{\sqrt{2}} \qquad f(x) = \sqrt{3}x^{-\pi}$

Power functions are fairly ubiquitous in natural systems, and in many examples, the power is not an integer. We will now work through a case study that showcases power functions in the context of socioeconomics.

4.1.1 Case Study: *The Rank-Size law of cities*

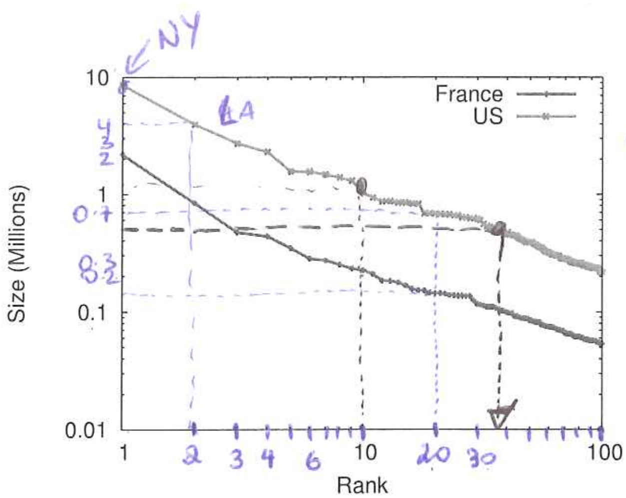
A remarkable empirical law of socio-economics is the observed relationship between the rank and size of the population of each city in a given country. To construct a graph showing this relationship, simply graph the population of a city against its rank (i.e. 1 for largest, 2 for second largest, 3 for third largest, etc..). Here are two examples, one for the US, and one for France:



- Sizes decreases with rank
- US cities are always bigger than french at the same rank

→ But otherwise the graphs don't look special

Unfortunately, the graph is not as informative as we would like, mostly because the population of the largest cities is so much larger than that of the smaller ones. An alternative way of plotting the data is to use a log-log scale. We will learn more about them in a few lecture's time, but just note for now how, on each axis, the numbers 1, 10, 100, etc. are equally-spaced, instead of the numbers 1,2,3,... being equally spaced. When using a log-log plot, something remarkable happens to this data:



- The data on this new set of axes lies almost in a straight line
- The line for US cities is nearly parallel to the line for french cities

As we will learn shortly, when data falls on a straight line in a log-log plot, this is symptomatic of a power-law relationship. It is in fact how scientists prove that a relationship is a power law. Let's try to fit the data with a function of the kind $s = f(r) = ar^b$, where s is the size, and r is the rank: we get

$S = ar^b$

Size is a function of rank as

US cities: $s(r) = \underline{7\ 10^6}\ r^{-0.75}$

French cities $s(r) = 1.3\ 10^6\ r^{-0.75}$

Using these two functions, we therefore see that the 10th largest cities in France and the US have population size of about

→ Plug in 10 for r in the formula

US cities: size is size $= 7 \times 10^6 \times 10^{-0.75}$
$= 1.2\ 10^6$ San José

French city size $= 1.3\ 10^6 \cdot 10^{-0.75} = 230{,}000$ Lille

We will now learn a little more about power laws, both in terms of where they occur, and what we can do with them.

4.1.2 More examples of power laws in nature

There are many examples of power laws in nature.

- Some of the most common ones are geometric relationships such as :

Volume of sphere as function of radius $V(r) = \frac{4}{3}\pi r^3$

Area of circle as function of radius $A(r) = \pi r^2$

Circumference of circle $C(r) = 2\pi r$

- Another common class of power laws are allometric laws in biology:

  - Allometric laws of body proportions
  - Kleiber's law of metabolism.

  Rate (kcal/day) $\simeq 70\,(\text{Mass in kg})^{0.75}$

- In fact, there are examples of power laws in most fields of Science!

  - Force of gravitation (Newton's law)

  $F(d) \propto \frac{1}{d^2}$ (Force of gravitation proportional to $\frac{1}{d^2}$, d = distance).

4.1.3 Manipulations of power functions

The following rules of exponents apply for manipulating power functions:

- $X^0 = 1$ for any x
- $x^1 = x$
- $x^a x^b = x^{a+b}$
- $x^{-a} = \frac{1}{x^a}$
- $\frac{x^a}{x^b} = x^{a-b} = x^a x^{-b}$
- $(x^a)^b = x^{ab}$

$2^2 \cdot 2^3 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^5 = 2^{2+3}$

$(2^3)^2 = 2^3 \cdot 2^3 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^6$

4.1.4 Graphs of power functions

The overall shape of the graph of a power function depends on the sign and value of the exponent b, as well as the number a of in front of course.

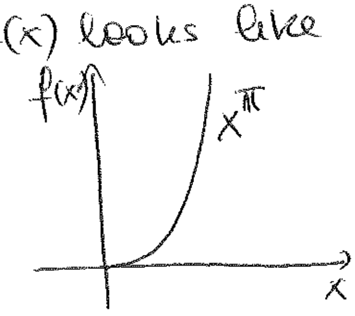
The graph of the function $f(x) = x^a$ depends on a in the following way

- If $\boxed{a < -1 \text{ or } a > 1}$ the graph looks like that of the integer power function with nearest power, for $\boxed{x > 0}$

Example $f(x) = x^{\pi}$

$\pi = 3.14\ldots \rightarrow$ nearest integer is 3

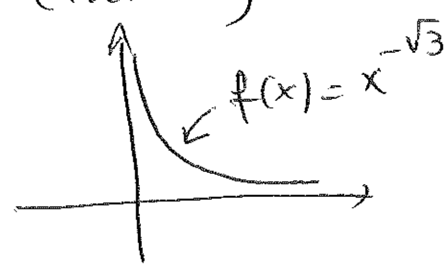
So $f(x)$ looks like x^3 (for $x > 0$)



$f(x) = x^{-\sqrt{3}}$

$-\sqrt{3} \simeq -1.7\ldots$

so $f(x)$ will look like x^{-2} (for $x > 0$)



- If $-1 < a < 1$ it's more complicated (see later)

4.2 Inverse of functions and Composition of functions

Textbook section 4.2

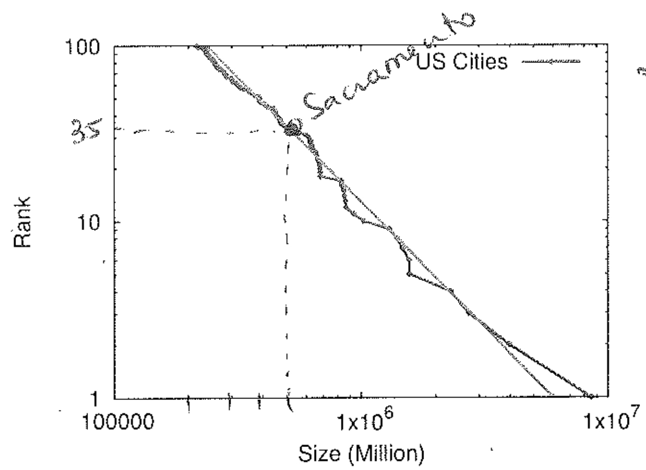
4.2.1 Case Study: *The Rank-Size law of cities*

Sacramento is the capital of California, and has a population size of about 490,000 people. What is its rank based on the graph?

look up 0.5 million on y-axis, draw straight line toward graph & read the x-value corresponding to point : 38

This is in fact a fairly good estimate, as the true rank of Sacramento is 35

We can apply the same technique to many cities on this graph, i.e. look up their rank based on their size. We could in fact create a graph with this information, which would allow us to retrieve it and share it with others much more efficiently. This would give the following plot (on a log-log scale):



→ This is equivalent to switching the x- and y-axes on the original graph.

→ This is the graph of a new function $r(s)$

Now, what about Santa Cruz? The population of Santa Cruz is roughly 70,000 people, so what is its rank? Unfortunately, the graph above is not very useful because it does not show what happens for cities of less than 200,000 people. So what can we do? The solution here is not to use the graph, but to use what we know about the Rank-Size relationship of US cities:

→ Use the fact that $S = 7.10^6 \, r^{-0.75}$ with $S = 70{,}000$ $\Rightarrow$ $70000 = 7000000 \, r^{-0.75}$ and solve for r

$$r^{-0.75} = \frac{\cancel{70000}}{7\cancel{000000}} = \frac{1}{100} \Rightarrow r = \left(\frac{1}{100}\right)^{\frac{1}{-0.75}} = 100^{\frac{1}{0.75}}$$

$\boxed{r \simeq 464}$

In fact, we could do this for any US city, namely, to get an estimate of the rank as a function of their size:

→ Solve $S = 7.10^6 \, r^{-0.75}$ for r, for any value of s

$$\rightarrow \frac{S}{7\,10^6} = r^{-0.75} \rightarrow \boxed{r = \left(\frac{S}{7\,10^6}\right)^{-\frac{1}{0.75}}}$$

→ This is the new function $r(S)$

In doing so, we have created another function $g(s)$, which takes the size and returns the rank: $r = g(s)$. If we plot this new function on the graph of Rank vs. Size, we see that it indeed fits the data very well. We can therefore use this, as an alternative option to the graph, to find the estimated rank of any city in the US knowing its size!

The functions f and g are clearly related to one another:

$$s = f(r) = 7.10^6\, r^{-0.75}$$
$$r \rightleftarrows s$$
$$r = g(s) = \left(\frac{s}{7\,10^6}\right)^{-0.75}$$

In fact, they are called Inverse of one another! We will now learn more about inverses, and generalize the concept we have just learned.

4.2.2 Definition of the inverse, and examples

DEFINITION: The inverse of a function $f(x)$ is another function, usually called $f^{-1}(x)$, which <u>reverses</u> the action of f

$$x \underset{f^{-1}}{\overset{f}{\rightleftarrows}} y$$

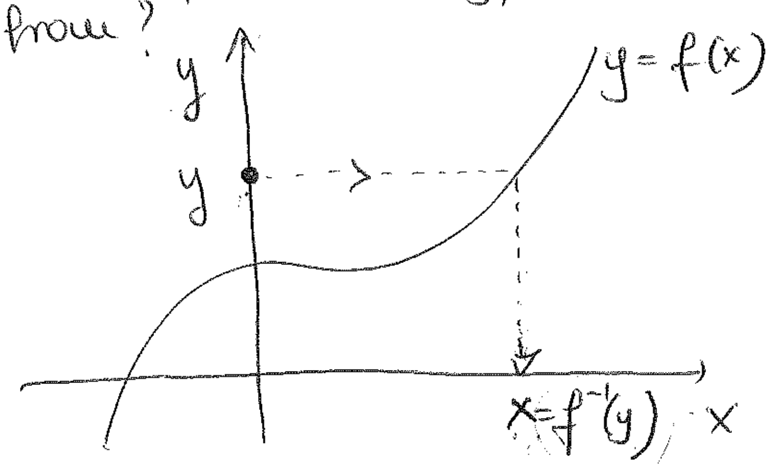
The functions f and f^{-1} satisfy

If $\boxed{y = f(x) \text{ then } x = f^{-1}(y)}$

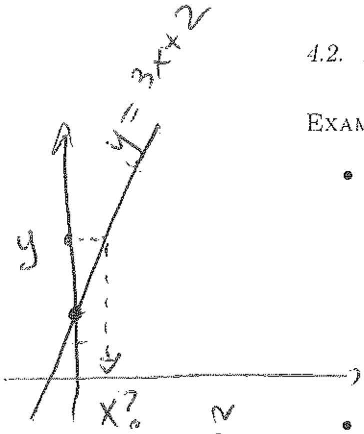
GRAPHICAL INTERPRETATION:

Graphically, the inverse function answers the question:

Given $y=f(x)$, if I know y, which x did it come from?



⇒ Finding the inverse of a function $f(x)$ boils down to solving $y=f(x)$ <u>for x</u>



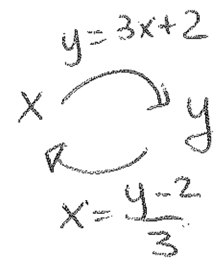
EXAMPLES

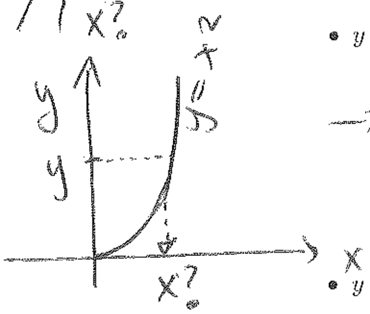
- $y = f(x) = 3x + 2$:

$\to$ take $y = 3x+2$ and solve for x

$\to$ $y-2 = 3x \to x = \frac{y-2}{3}$

$x = \boxed{f^{-1}(y) = \frac{y-2}{3}}$

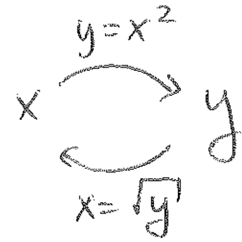




- $y = f(x) = x^2$ (for $x \geq 0$):

$\to$ take $y = x^2$ and solve for x

$\to$ $x = \sqrt{y} = f^{-1}(y)$



- $y = f(x) = \sqrt{x-2}$ (for $x \geq 2$):

$\to$ take $y = \sqrt{x-2}$ and solve for x

$\to$ $y^2 = x-2 \to y^2+2 = x = f^{-1}(y)$

$f^{-1}(y) = y^2+2$ $\boxed{f^{-1}(x) = x^2+2}$

- $y = f(x) = \frac{3x+1}{x-2}$

$y = \frac{3x+1}{x-2} \to$ solve for x

$\to$ $y(x-2) = 3x+1 \to yx - 2y = 3x+1$

$\to$ $yx - 3x = 2y+1 \to x(y-3) = 2y+1$

$\to$ $x = \boxed{\frac{2y+1}{y-3} = f^{-1}(y)}$

$\to$ $\boxed{f^{-1}(x) = \frac{2x+1}{x-3}}$

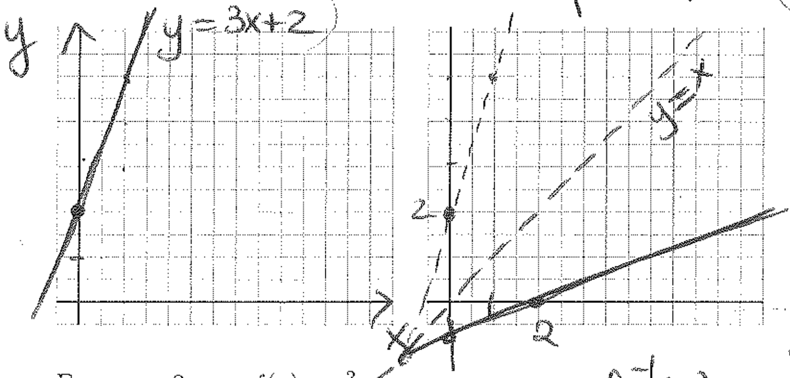
IMPORTANT NOTES:

f^{-1} is just a notation $\Rightarrow$
- $f^{-1}(x) \neq \frac{1}{f(x)}$
- If f^{-1} is the inverse of f
- then f is the inverse of f^{-1} $\to$ inverses come in pairs

4.2.3 Graph of an inverse function and horizontal line test:

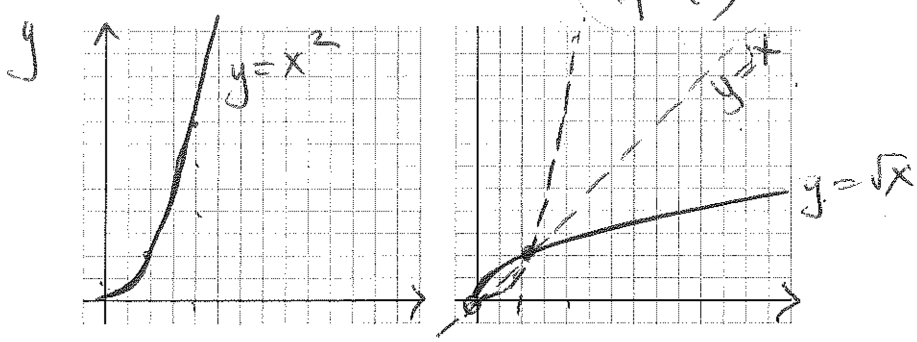
EXAMPLE 1: $y = f(x) = 3x + 2$

$f^{-1}(x) = \frac{x-2}{3}$



EXAMPLE 2: $y = f(x) = x^2$

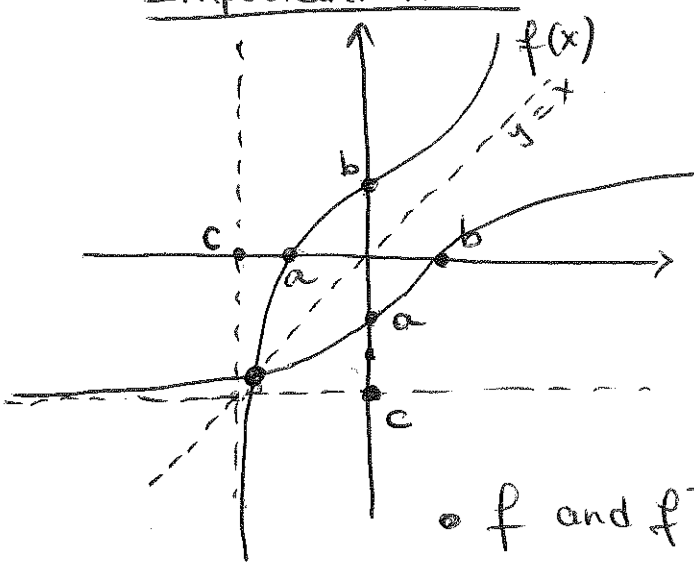
$f^{-1}(x) = \sqrt{x}$



So from these graphs we notice that:

The graph of the function $y = f^{-1}(x)$ is the mirror-image of the graph of the function $y = f(x)$ with respect to the $y = x$ line.

Important notes



- $f(a) = 0$ $\quad f(0) = b$

  f has vertical asymptote at c

$\Rightarrow$

- $f^{-1}(b) = 0$ $\quad f^{-1}(0) = a$

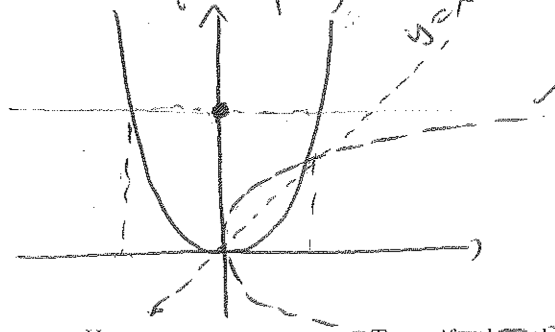
  root at b $\quad$ y-intercept at a

  f^{-1} has a horizontal asymptote at c

- f and f^{-1} only ever cross <u>on</u> the $y = x$ line

NOTE: It may happen that through this process, the graph of the inverse does not satisfy the vertical line test: in that case, the inverse is not defined.

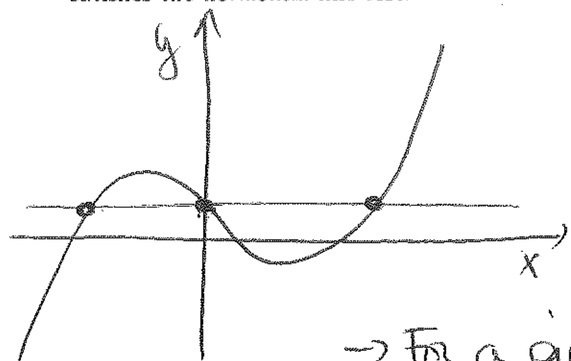
Example: $f(x) = x^2$



→ This graph is NOT the graph of a function, it does not pass the vertical line test

→ The inverse does not exist.

HORIZONTAL LINE TEST: ~~To verify that~~ the inverse of a function is unique, we check that the function satisfies the horizontal line test:



This function has no inverse

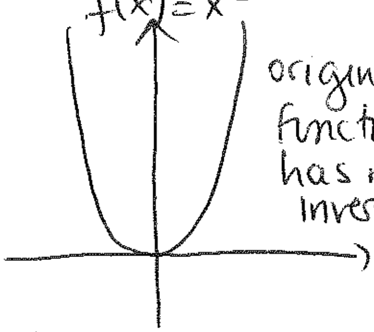
It does not pass the horizontal line test, ie, a line crosses the graph of $y = f(x)$ more than once.

→ For a given y, there are more than one x such that $y = f(x)$

When a function $f(x)$ does not satisfy the horizontal line test, we can often choose a smaller domain for which the inverse *is* unique.

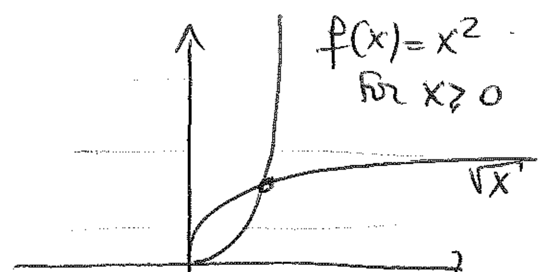
EXAMPLE: for the function $f(x) = x^2$, we saw earlier that the inverse of $f(x) = x^2$ is defined provided we select only the interval for which $x \geq 0$. In this interval, the function $f(x)$ does satisfy the horizontal line test.

$f(x) = x^2$



original function has no inverse

chop out a bit so it does pass the test.



- This passes the horizontal line test
- This function has an inverse.

$$y = f(x) \iff x = f^{-1}(y)$$

4.2.4 Composition of functions

Knowing that if $y = f(x)$, then $x = f^{-1}(y)$, we can come to a rather interesting conclusion:

$$y = f(x) = f(f^{-1}(y)) = f \circ f^{-1}(y)$$

$$x = f^{-1}(y) = f^{-1}(f(x)) = f^{-1} \circ f(x)$$

While this may have seemed to be a simple game of plugging one thing into another, the notion of applying a function to another function is actually a very important mathematical concept, called *the composition of two functions.*

Given any two functions $f(x)$ and $g(x)$, we can construct another function $h(x)$ as $h(x) = f(g(x)) \to$ the <u>composition</u> of f and g, also written as $f \circ g(x)$

$$\Rightarrow \quad f(g(x)) = \underbrace{f \circ g(x)}_{\text{new notation}}$$

EXAMPLES:

- $f(x) = \sin(x)$, $g(x) = 4x - 1$: $f \circ g$:

$$f \circ g(x) = f(g(x)) = f(4x-1) = \sin(4x-1)$$

- $f(x) = \frac{1}{x^2-2}$, $g(x) = x + 1$: $f \circ g$:

$$f \circ g(x) = f(g(x)) = f(x+1) = \frac{1}{(x+1)^2 - 2}$$

- $f(x) = \sqrt{1-x}$, $g(x) = x^2$: $g \circ f$:

$$g \circ f(x) = g(f(x)) = g(\sqrt{1-x}) = [\sqrt{1-x}]^2 = |1-x|$$

IMPORTANT NOTE: Changing the order of the composition yields an entirely different function!

EXAMPLE: $f(x) = \sqrt{x}$, $g(x) = x^2 + 1$ $\quad f = \sqrt{x} \quad g = x^2 + 1$

- $f \circ g$:

$$f \circ g(x) = f(g(x)) = f(x^2+1) = \sqrt{x^2+1}$$

- $g \circ f$:

$$g \circ f(x) = g(f(x)) = g(\sqrt{x}) = (\sqrt{x})^2 + 1 = |x| + 1$$

4.2.5 The composition of a function and its inverse

As we found out above, there are two fundamental relationships between a function and its inverse:

f and f^{-1} cancel each other out.

$$\begin{cases} f(f^{-1}(x)) = x & \text{for any } x \\ f^{-1}(f(x)) = x & \text{for any } x \end{cases}$$

We can check that this is true in all the examples we have seen before:

- $f(x) = 3x + 2 \qquad f^{-1}(x) = \frac{x-2}{3}$

$$f(f^{-1}(x)) = f\left(\frac{x-2}{3}\right) = \cancel{3} \cdot \frac{x-2}{\cancel{3}} + 2 = x - \cancel{2} + \cancel{2} = x \checkmark$$

$$f^{-1}(f(x)) = f^{-1}(3x+2) = \frac{3x + \cancel{2} - \cancel{2}}{3} = \frac{\cancel{3}x}{\cancel{3}} = x \checkmark$$

- $f(x) = \frac{3x+1}{x-2} \qquad f^{-1}(x) = \frac{2x+1}{x-3}$

$$f(f^{-1}(x)) = f\left(\frac{2x+1}{x-3}\right) = \frac{3\,\frac{2x+1}{x-3} + 1}{\frac{2x+1}{x-3} - 2}$$

$$= \frac{\frac{6x+3}{x-3} + \frac{x-3}{x-3}}{\frac{2x+1}{x-3} - \frac{2(x-3)}{x-3}}$$

$$= \frac{\frac{6x + \cancel{3} + x - \cancel{3}}{x-3}}{\frac{\cancel{2x} + 1 - \cancel{2x} + 6}{x-3}} = \frac{\frac{7x}{x-3}}{\frac{7}{x-3}}$$

$$= \frac{\cancel{7}x}{\cancel{x-3}} \cdot \frac{\cancel{x-3}}{\cancel{7}} = x$$

Inverses of power functions and their graphs

- Given the function $f(x) = x^{a}$, the inverse is $f^{-1}(x) = x^{\frac{1}{a}}$.

<u>Check</u>: $f(f^{-1}(x)) = f\left(x^{\frac{1}{a}}\right) = \left(x^{\frac{1}{a}}\right)^{a} = x^{\frac{a}{a}}$
$= x^{1} = x$ ✓

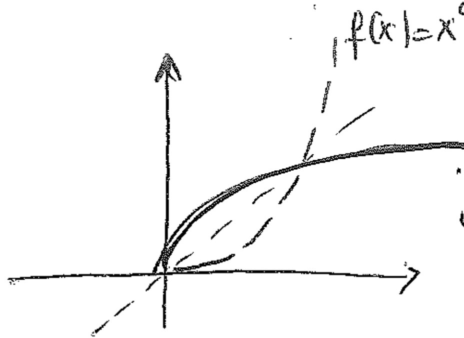
<u>Example</u>:
- The inverse of $f(x) = x^2$ is $f^{-1}(x) = x^{\frac{1}{2}} = \sqrt{x}$
- The inverse of $f(x) = x^3$ is $f^{-1}(x) = x^{\frac{1}{3}}$.

→ This can be used to find the graph of power function with exponents between -1 and 1

<u>Recap</u>:

If $f(x) = x^{a}$, $a > 1$

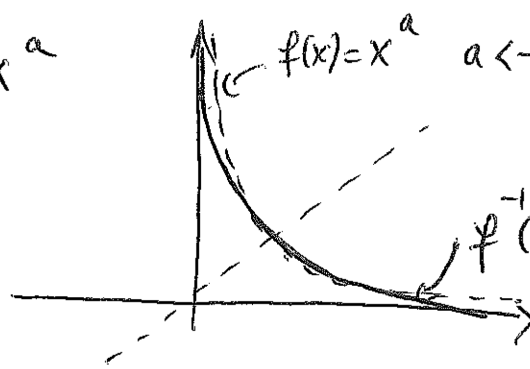
$f^{-1}(x) = x^{\frac{1}{a}}$, $a > 1$



↳ this is the graph for any function $x^{\frac{1}{a}}$ where $a > 1$
example: $x^{1/2}$, $x^{1/\pi}$, etc...

If $f(x) = x^{a}$, $a < -1$

$f^{-1}(x) = x^{\frac{1}{a}}$, $a < -1$



↳ this is the graph for <u>any</u> function of this kind e.g: $x^{-\frac{1}{2}}$ or $x^{-\frac{1}{\pi}}$