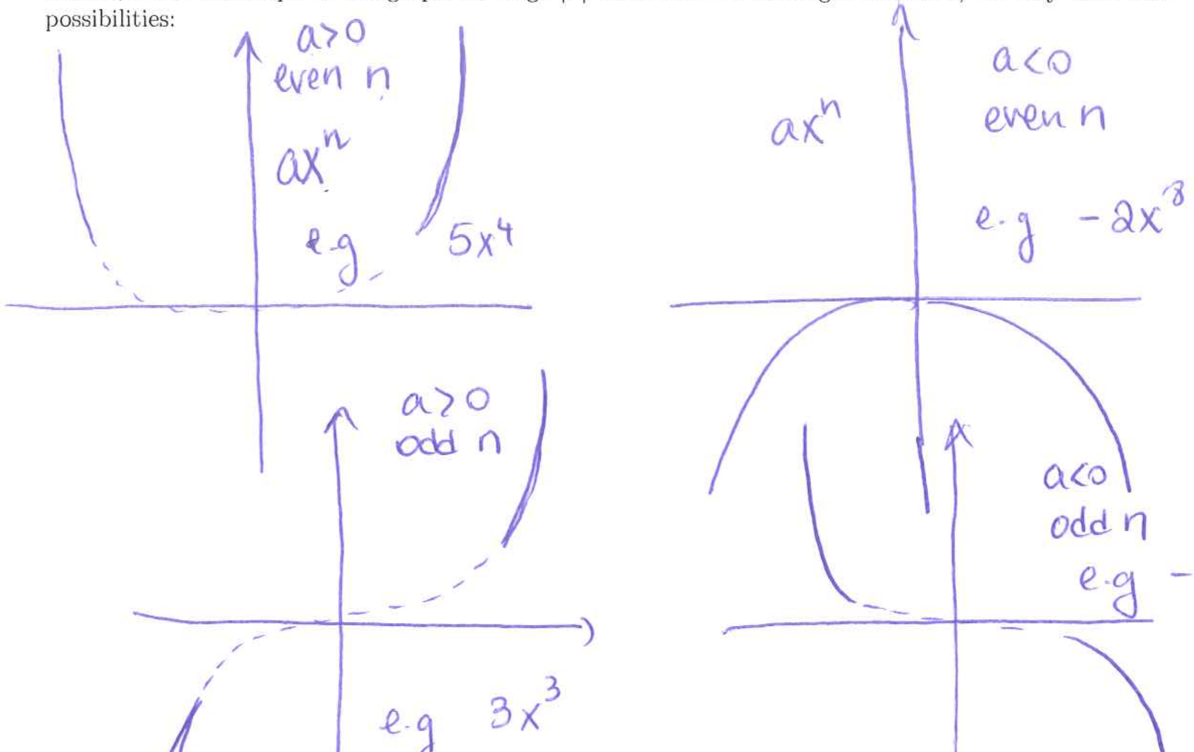
Chapter 3

Rational functions

Until now, we have learned about a general class of functions called polynomial functions, that include linear functions (i.e. polynomials of order 1) and quadratic functions (i.e. polynomials of order 2), as well as higher order polynomials. We saw that the graphs of linear functions are always straight lines, while the graphs of quadratic functions are always parabolas. The graphs of higher order polynomials are more diverse, but always either go up to $+\infty$ or down to $-\infty$ as x becomes large (either positive or negative). Indeed, since the shape of the graph for large $|x|$ looks like the leading order term, we only have four possibilities:

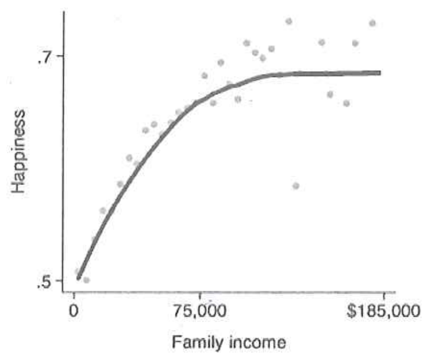


On the other hand, there are many situations in which we would like to create a function to model data, or to model an idea, where the dependent variable does *not* go to $\pm\infty$. Here are a number of examples.

3.1 Case study: *Modeling functions that are not polynomials*

3.1.1 Happiness as a function of income

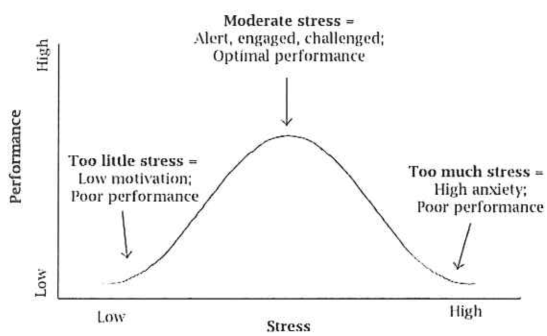
The following data is extracted from the General Social Survey website by Lane Kenworthy, in a book called 'The Good Society'. It shows a self-reported estimate of people's happiness across the US as a function of income from surveys conducted between 1972 and 2012. Happiness is quantified by the responders (about 50,000 of them!), with 0 meaning 'not too happy', 0.5 meaning 'pretty happy' and 1 meaning 'very happy'.



- This function {tends to / asymptotes to} a constant as the independent variable tends to $+\infty$

3.1.2 The effect of stress on performance

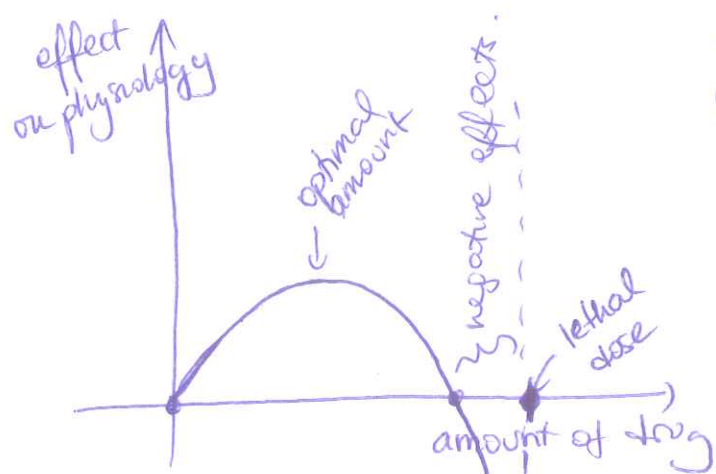
The Yerkes-Dodson effect in psychology is an old empirical relationship put forward after a series of experiments on rats in the early 1900s. Yerkes and Dodson argued from their experiments that the rats' performance in various tasks increased with the level of stress they were under, but only up to a point, after which the performance started to decrease again. This law is often discussed in manageurial psychology as a possible way of getting workers to perform better (i.e. don't make them too comfortable). There is a lot of debate about the validity of this law, with many still using it for manageurial purposes, and others arguing that it should long have been discredited.



- This function {tends to / asymptotes to} zero as the independent variable tends to $+\infty$.

3.1.3 The effect of drugs

Many drugs which are designed to have a beneficial health effect can become extremely toxic, and even deadly, when taken in large quantities. This change from beneficial to toxic to lethal can be modeled with the following function:



- This function tends to $(-\infty)$ as the independent variable approaches the lethal dose (i.e a particular value of the independent variable)

3.1.4 The need for more flexible functions

These examples illustrate the great diversity of functions required to model real life-examples, and this diversity simply cannot be captured by polynomials alone. For this reason, we will now learn about a greater class of functions, of which polynomials are a mere subset, and that have a much greater diversity in behavior. These functions are called Rational Functions.

3.2 Rational functions

Textbook section 3.4-3.5

3.2.1 General definition and properties of rational functions

DEFINITIONS: A rational function is the ratio of two polynomials

→ There are two forms

- The expanded form: $f(x) = \frac{a_0 + a_1x + a_2x^2 + \dots a_nx^n}{b_0 + b_1x + \dots + b_mx^m}$
  (where m & n need not be identical)
- The factored form

$$f(x) = \frac{(x - x_1) \dots (x - x_r) q_1(x)}{(x - x_{r+1}) \dots (x - x_d) q_2(x)}$$

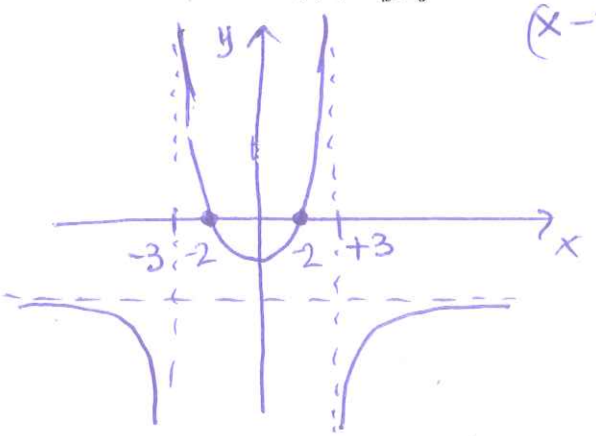
Examples $f(x) = \frac{3x^2 - 2x + 1}{x - 2}$

or $f(x) = \frac{x(x+2)(x^2+x+1)}{2(x-3)(x+4)}$

Domain of definition : The domain excludes all the roots of the denominator (the denominator cannot be zero).

Asymptotes: When the denominator of the rational function $f(x)$ goes to zero, the graph of $y = f(x)$ usually has a vertical asymptote, as in the following example:

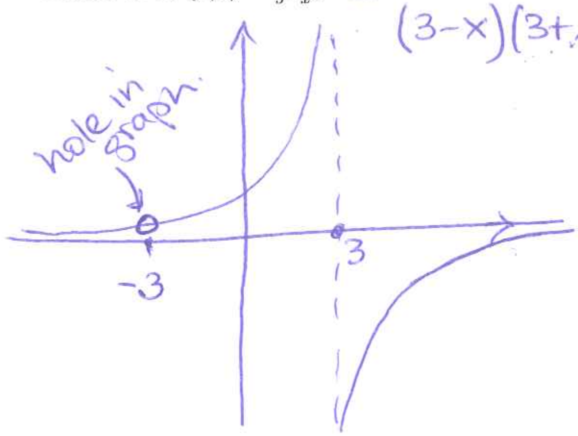
Example 1: $f(x) = \frac{4-x^2}{x^2-9}$ $= \frac{4-x^2}{(x-3)(x+3)}$ → 2 roots in the denominator $x=3$ and $x=-3$



- Instead of having values for $f(3)$ and $f(-3)$, the function does not exist at these points (not in domain)
- The graph avoids them, and instead tends to (but never crosses) a vertical line (one at $x=-3$, one at $x=3$)

However, it may happen that the asymptote is "canceled out" by a root in the numerator. This situation is easy to determine from the factored form of $f(x)$. When this is the case, the function must be simplified first!

Example 2: $f(x) = \frac{x+3}{9-x^2}$ $= \frac{x+3}{(3-x)(3+x)} = \frac{1}{3-x}$ unless $x=-3$



- The graph has a hole at $x=-3$ to remind ourselves that $x=-3$ is not in the domain of f.

3.2.2 The behavior of rational functions for large $|x|$

In order to study the behavior of rational functions for large $|x|$ (that is, x going to $+\infty$ or x going to $-\infty$), we use the property learned in the previous chapter about the behavior of *polynomial* functions for large $|x|$:

$f(x) = a_0 + a_1x + \dots a_nx^n$ looks like the leading term a_nx^n for large x.

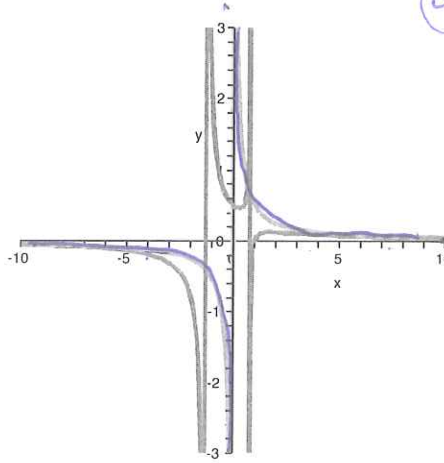
As a result, for rational functions,

We can therefore see that rational functions can have many different kinds of behavior, depending on the order of the polynomials in the numerator relative to that of the denominator.

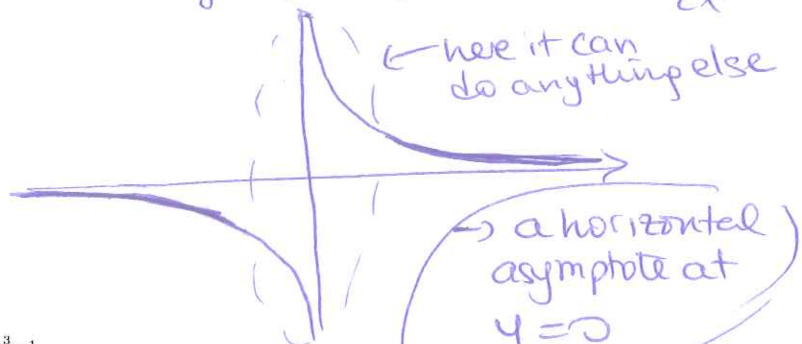
$f(x) = \frac{a_0 + a_1x + \dots a_nx^n}{b_0 + b_1x + \dots b_mx^m}$ look like $\frac{a_nx^n}{b_mx^m}$

for large x

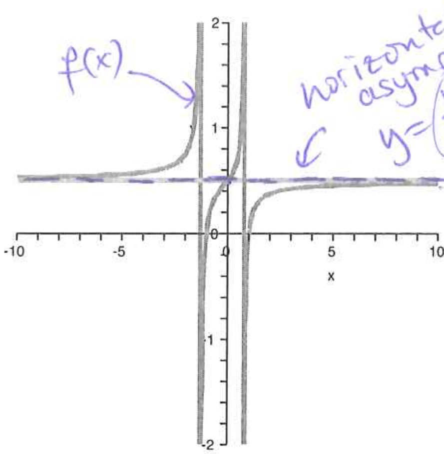
EXAMPLE 1: $f(x) = \frac{x-1}{2x^2+x-2}$ $= \frac{x-1}{2x^2+x-2}$ looks like $\frac{x}{2x^2} = \frac{1}{2x}$

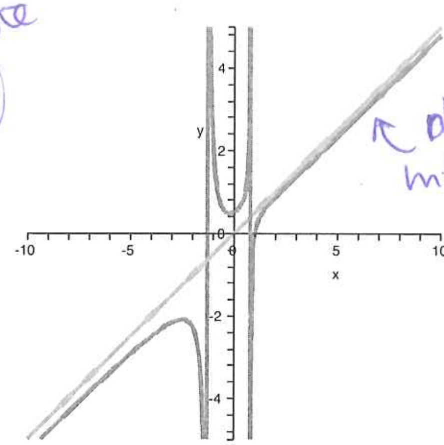


For large x, $f(x)$ looks like $\frac{1}{2x}$



EXAMPLE 2: $f(x) = \frac{x^2-1}{2x^2+x-2}$, $g(x) = \frac{x^3-1}{2x^2+x-2}$



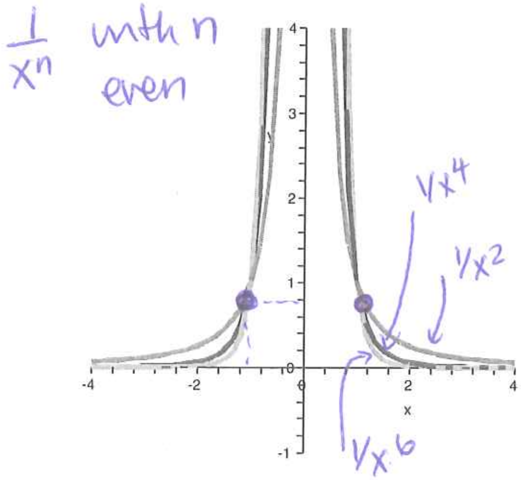


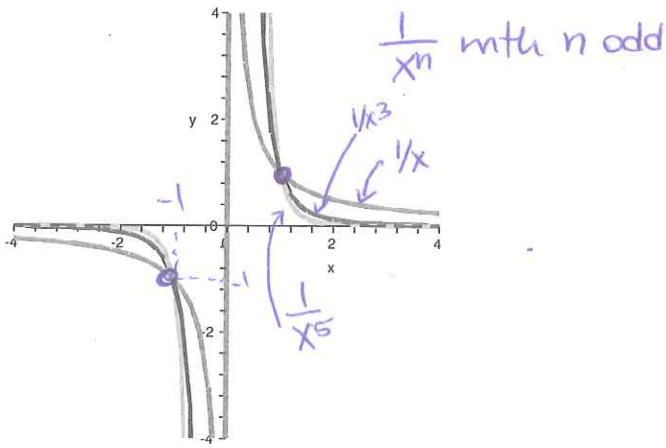
↖ oblique asymptote with equation $y = \frac{x}{2}$

$\frac{x^2-1}{2x^2+x-2}$ looks like $\frac{x^2}{2x^2} = \frac{1}{2}$

$\frac{x^3-1}{2x^2+x-2}$ looks like $\frac{x^3}{2x^2} = \frac{x}{2}$

As a first step towards understanding the behavior of rational functions for large $|x|$, we therefore have to remember what the graph of functions of the kind $f(x) = x^{-n}$. As in the case of power functions, we have two different kinds of behavior depending on whether the power n is even or is odd:





As in the case of power functions with positive integer powers, the function $f(x) = x^{-n}$ is even if n is even, and is odd if n is odd:

NOTE:

- For all of the graphs, $x=0$ is a vertical asymptote.
- $f(1) = 1$ $\quad \frac{1}{(1)^n} = 1$ $\quad \frac{1}{(-1)^n} = \begin{cases} 1 & \text{if } n \text{ even} \\ -1 & \text{if } n \text{ odd} \end{cases}$

3.2.3 Studying rational functions using signs tables

Signs tables are extremely useful tools for studying rational functions. They are used in nearly exactly the same way as for polynomial functions:

- Cast the function in a fully factored form, for both the numerator and the denominator. Simplify as needed before proceeding.
- Draw the table
- Write **all** the factors vertically on the left, including both the numerator and the denominator.

- Write **all** the points where either the numerator or the denominator goes to 0 on the top, in the correct order. Draw vertical lines below each of them.
- Determine and write the sign of each factor; write zeros where there is a root, and an infinity sign where there is an asymptote.
- Multiply the signs in each interval to determine the sign of the function.

The advantage of this method is that it can tell you very easily what the behavior of rational functions near an asymptote is, and helps graph it. We can also combine it with the information we obtained from the behavior as $|x|$ tends to infinity.

EXAMPLE 1: Study and sketch the function $f(x) = \frac{x}{x^2-4}$

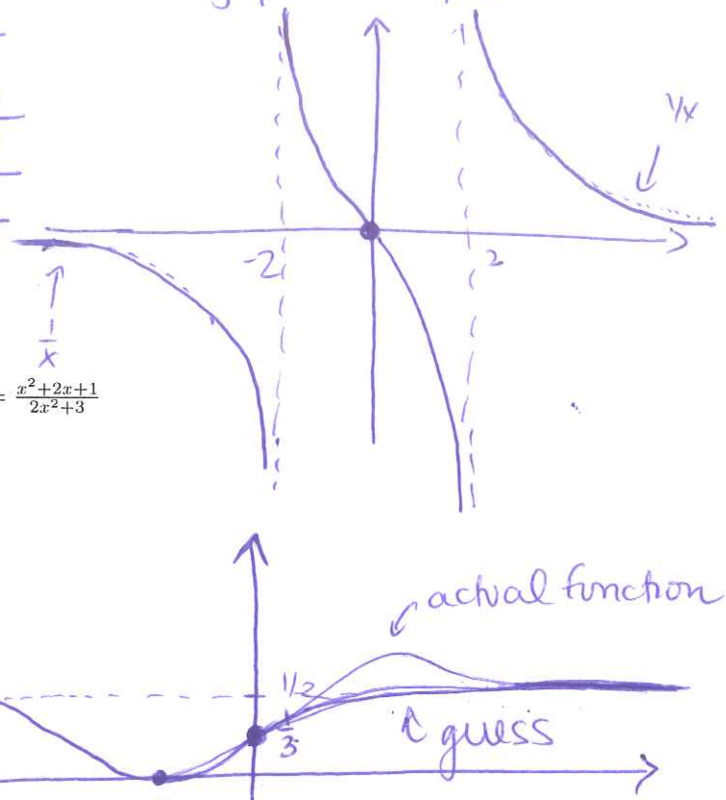
$\frac{x}{x^2-4} = \frac{x}{(x-2)(x+2)}$

1 root (0)

2 asymptotes −2, 2

		−2		0		2	
x	−		−	0	+		+
x−2	−		−		−	∞	+
x+2	−	∞	+		+		+
	−	∞	+	0	−	∞	+

Behavior at ∞ : $f(x) \to \frac{x}{x^2} = \frac{1}{x}$



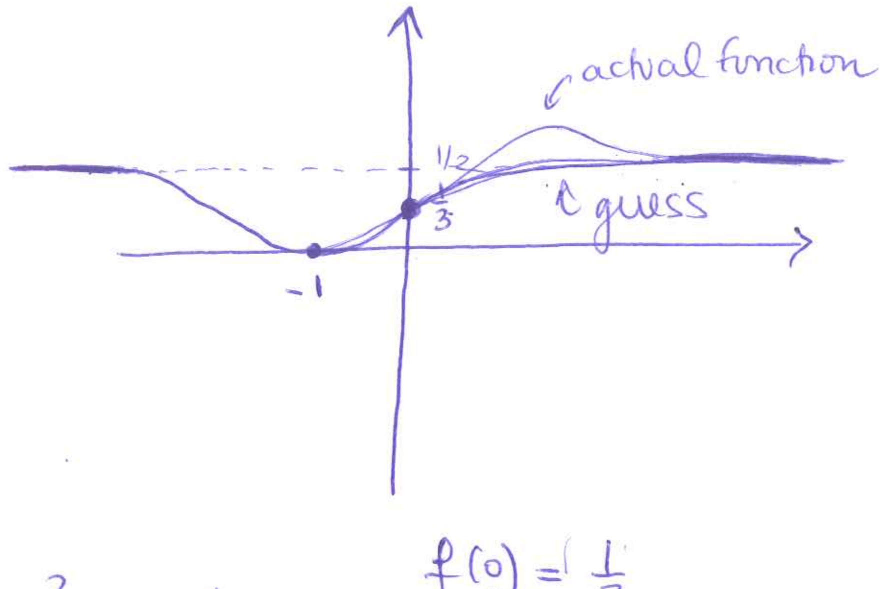
EXAMPLE 2: Study and sketch the function $f(x) = \frac{x^2+2x+1}{2x^2+3}$

$\frac{x^2+2x+1}{2x^2+3} = \frac{(x+1)^2}{2x^2+3}$

one root (−1)

no asymptote

		−1	
(x+1)	−	0	+
(x+1)	−	0	+
$2x^2+3$	+		+
	+	0	+



$f(0) = \frac{1}{3}$

Behavior at ∞ : $f(x) \to \frac{x^2}{2x^2} = \frac{1}{2}$.

EXAMPLE 3: Study and sketch the function $f(x) = \frac{x^3+2x^2-3x}{x^2-2x+1}$

$$x^3+2x^2-3x = x(x^2+2x-3)$$

$$D = 2^2-4(1)(-3) = 4+12=16$$

$$x_{1,2} = \frac{-2\pm\sqrt{16}}{2(1)} = \frac{-2\pm 4}{2} = \begin{cases} 1 \\ -3 \end{cases}$$

$$x^3+2x^2-3x = x(x-1)(x+3)$$

$$x^2-2x+1 = (x-1)^2$$

$$f(x) = \frac{x(x-1)(x+3)}{(x-1)^2} = \frac{x(x+3)}{x-1} \text{ if } x \neq 1$$

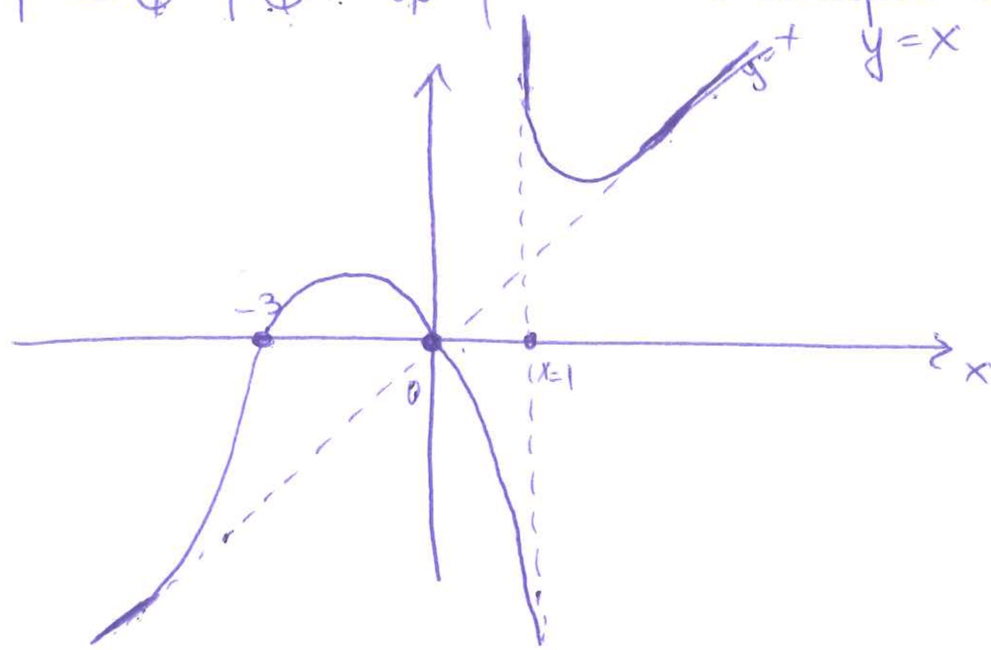
if $x=1$, ~~hole in graph~~ asymptote

		-3		0		1	
x	$-$		$-$	0	$+$		$+$
$x+3$	$-$	0	$+$		$+$		$+$
$x-1$	$-$		$-$		$-$	0	$+$
	$-$	0	$+$	0	$-$	∞	$+$

- $f(0) = 0$
- Behavior for large x:

$$f(x) = \frac{x^3}{x^2} = x$$

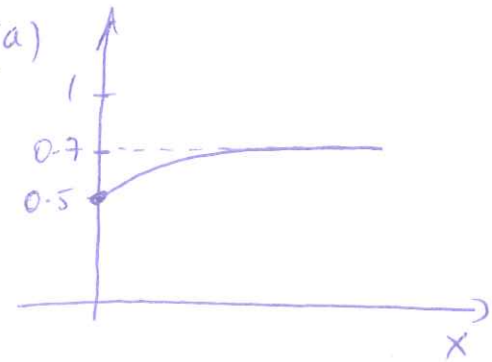
→ oblique asymptote $y = x$



3.3 Case study: *Modeling functions that are not polynomials*

We can now go back to our case study and try to reverse-engineer what kind of function may have a graph that looks like the data or the idea we want to model .

(a)



$f(x)$ has properties

- $f(0) = 0.5$
- $f(x)$ has a horizontal asymptote at $y = 0.7$

- To get horizontal asymptote for large x, we need the ratio of the leading terms in numerator & denominator to be equal to 0.7

⇒ the order of these two polynomials have to be the same.

→ let's take simple linear functions

$$f(x) = \frac{0.7x + a}{x + b} \quad a, b \text{ just numbers.}$$

- To get $f(0) = 0.5$, we need

$$f(0) = \frac{a}{b} = 0.5$$

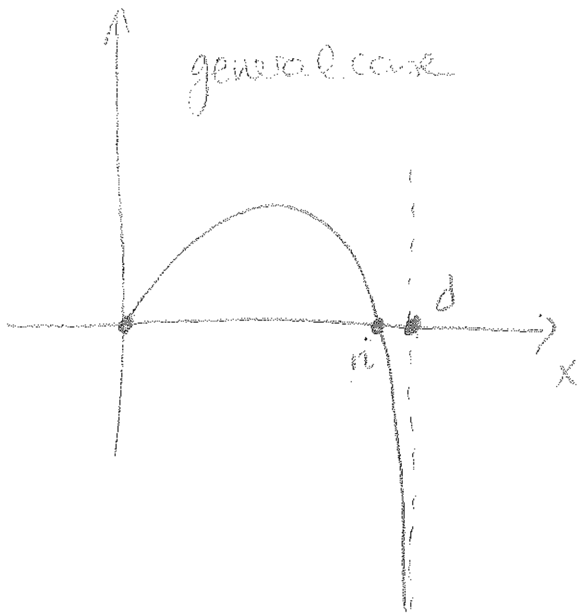
→ we can pick any a & b as long as their ratio is 0.5, for example

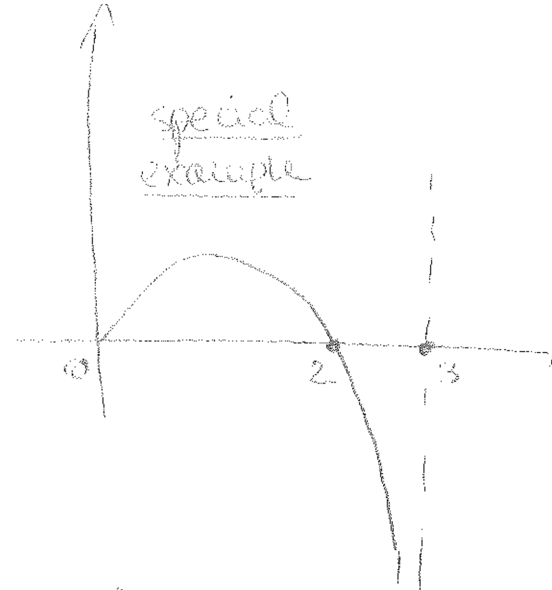
$a = 0.5$ and $b = 1$

$a = 5000$ and $b = 10,000$

→ Try $f(x) = \dfrac{0.7x + 0.5}{x + 1}$

(c)





properties needed

- $f(0) = 0$
- $f(n)$ or $f(2)$ are 0
- vertical asymptote at $x = d$
  $x = 3$

Try: $f(x) = \dfrac{x(x-2)}{x-3}$ for example

$$f(x) = \frac{x(x-n)}{x-d} \text{ in general.}$$