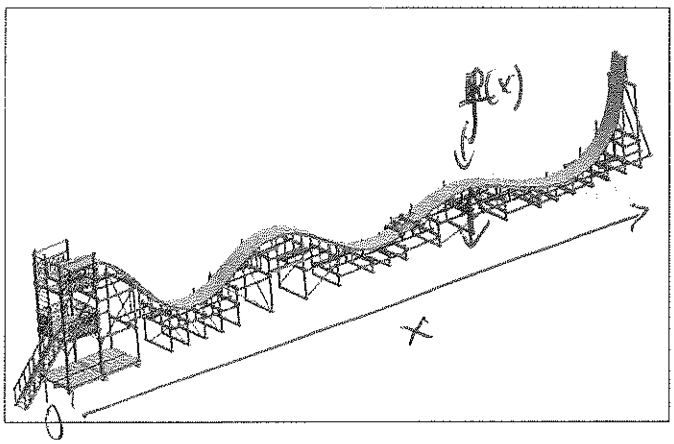
2.3 Higher-order polynomials

Textbook sections 3.1-3.2

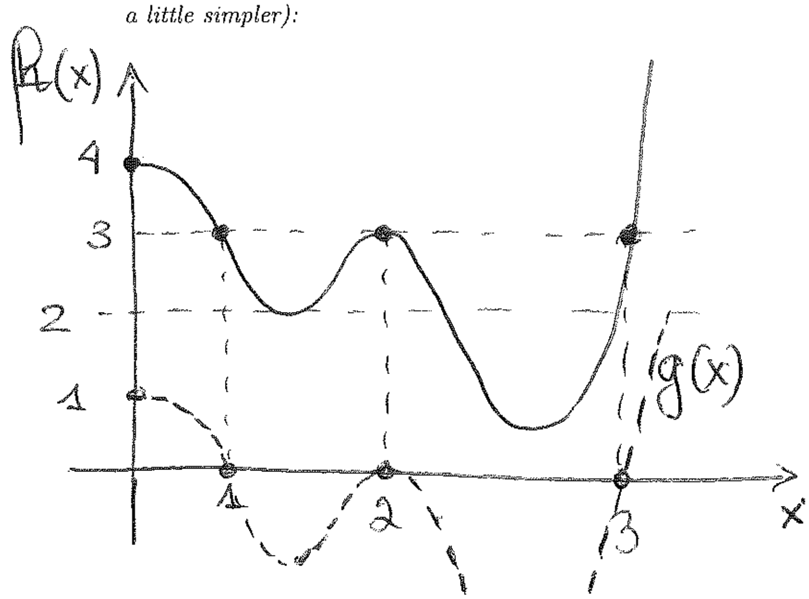
2.3.1 Case study: *How to build a roller-coaster*

Every year, MIT engineering students get to design, build and then test a real life wooden roller coaster as part of one of their design projects, in what is now known as the Easter Campus Roller Coaster. The design project involves many parts, including selecting the shape of the roller-coaster, and modeling the acceleration, velocity and trajectories of the cars (to make sure they don't fly off or are hazardous to the riders). To do so, students must first start by selecting a function that describes the height of the roller coaster as a function of the distance from the start. Let's look at the design the students selected:



h(x) is height of roller-coaster above ground.

While modeling this function may seem daunting at first (so many wiggles!) it turns out that it is not too difficult. Let's first draw an abstract version of this shape, with one less wiggle (just to make the problem a little simpler):



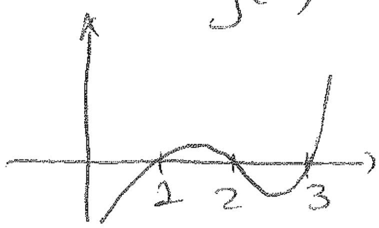
f(x) can be quite hard to model, but g(x) will actually be easy

While you may be wondering why is it easier to model the function $g(x) = f(x) - 3$ instead of the function $f(x)$ itself, note that $g(x)$ has a few interesting properties:

$g(x)$ has zeroes at $x=1$, $x=2$ and $x=3$

There is a very simple way of writing out a function that is zero at three specific points, and that involves writing it out as a product of three functions, each of which is equal to zero at one of the points. Based on that, what might be a good guess for $g(x)$ (and therefore also $f(x)$?)

$g(x)$ might be $(x-1)(x-2)(x-3)$?



pros: Got zeroes right ✓
It wiggles ✓
It has right behavior for $x>3$

But not quite right.

As it turns out, this guess isn't quite right, but at least it puts us in the right direction. Indeed, The function $g(x)$ that we guessed here is called a polynomial function, and as it turns out, the true answer is indeed a polynomial (of a slightly different form). Let's now learn more about polynomials, and then get back to this problem.

2.3.2 Definition and examples

DEFINITION: A polynomial function in expanded form is a function of the kind $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots + a_n x^n$
$a_0, a_1, \dots a_n$ are real numbers.

VOCABULARY:

- n is called the order of the polynomial
- $a_n x^n$ is called the leading-order term.

EXAMPLES: $f(x) = 4x^2 - 3x + 1$ order is 2 lead coeff is $4x^2$

$g(y) = 1 + 3y^2 - 5y^6 + y^9$ order is 9 lead coeff is y^9

$f(z) = 3x^2 + 2xz$ order is 1 leading term is $2xz$

In the case study above, our guess for the function $g(x)$ was indeed a polynomial, although it is not necessarily obvious from the formula we came up with. To check that it is, we can simply expand it.

$$(x-1)(x-2)(x-3) = (x-1)(x^2 - 3x - 2x + 6) = (x-1)(x^2 - 5x + 6)$$
$$= x^3 - 5x^2 + 6x - x^2 + 5x - 6$$
$$= x^3 - 6x^2 + 11x - 6 \rightarrow \text{order } 3$$

leading term is x^3

We therefore see that, as in the case of quadratics, there are two forms for a polynomial: the expanded form given earlier, and the factored form. Let's defined the factored form more precisely.

FORMAL DEFINITION OF FACTORED FORM: A fully factored polynomial is a function of the form $f(x) = (x - x_1)(x - x_2) \cdots (x - x_m) q(x)$

- where $x_1, x_2 \ldots x_m$ are the m roots of the polynomial (they need not be distinct),
- where $m \leq n$ (n is the order of $f(x)$)
- $q(x)$ is a polynomial with no roots ($q(x) \neq 0 \ \forall x$) for all

It is not always easy to determine whether a polynomial is fully factored, or can be factored further. Sometimes, the polynomial is already obviously fully factored. Sometimes, it is partially factored, and one must decide if the remaining part can be factored further or not. Sometimes the polynomial is fully expanded, and one must start factoring it from scratch.

EXAMPLES:

- $f(x) = -(2 + x)(x + 3)^3$ roots: -2, $\underbrace{-3, -3, -3}_{\text{(multiple)(repeated) root}}$

  Fully factored. $q(x) = -1$ $q(x) \neq 0$

- $f(x) = (x - 1)(2 - x^2) = (x-1)(\sqrt{2} - x)(\sqrt{2} + x) = (-)(x-1)(x-\sqrt{2})(x+\sqrt{2})$

  roots: $1, \sqrt{2}, -\sqrt{2}$

  $q(x) = -1$

- $f(x) = -2x(x^2 - 2x + 1)(x + 3) = -2x(x-1)^2(x+3) = -2(x-0)(x-1)^2(x+3)$

  roots: $0, 1, 1, -3$ $q(x) = -2$

- $f(x) = x^3 + 2x^2 + 4x = x\underbrace{(x^2 + 2x + 4)}_{\substack{\text{cannot be factored} \\ \text{no roots} \\ = q(x)}}$ $D = 2^2 - 4(1)(4) = -12$

  1 root: 0

In the examples presented above, it is still reasonably easy to factor the polynomial, either by finding a common factor, or by recognizing one of the standard patterns. However, there are many case in which it is not so easy. In fact, factoring high-order polynomials is notoriously difficult, and in some cases can only be done numerically.

Once we have both the expanded and fully factored forms of a polynomial function, we can learn a lot about its graph. For instance, from the expanded form we can deduce what the graph looks like for large and small x, just as we did for quadratics.

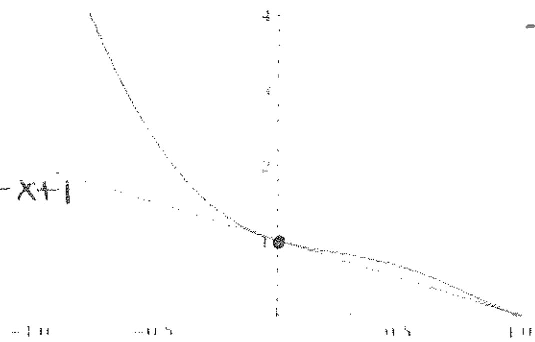
APPROXIMATIONS OF POLYNOMIALS FOR VERY SMALL VALUES OF x (near the y-axis)

When x is very small, the polynomial $f(x) = \underbrace{a_0 + a_1 x}{} + \cdots a_n x^n$ looks like the graph of $y = a_0 + a_1 x$

EXAMPLE: $f(x) = x^5 - 3x^3 + 2x^2 \underbrace{- x + 1}{}$

The straight line $y = -x + 1$

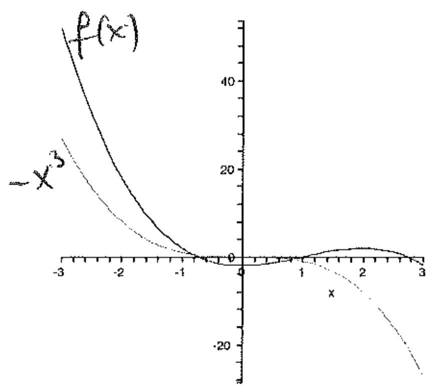
$y = -x + 1$

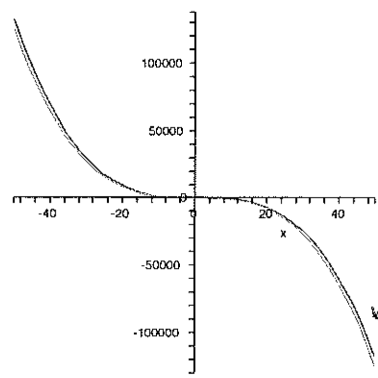


APPROXIMATIONS OF POLYNOMIALS FOR VERY LARGE VALUES OF $|x|$

When $|x|$ is very large, then the polynomial $f(x)$ is well approximated by leading-order term $a_n x^n$

EXAMPLE: $f(x) = -x^3 + 3x^2 - 2$ is very similar to $-x^3$ for large $|x|$



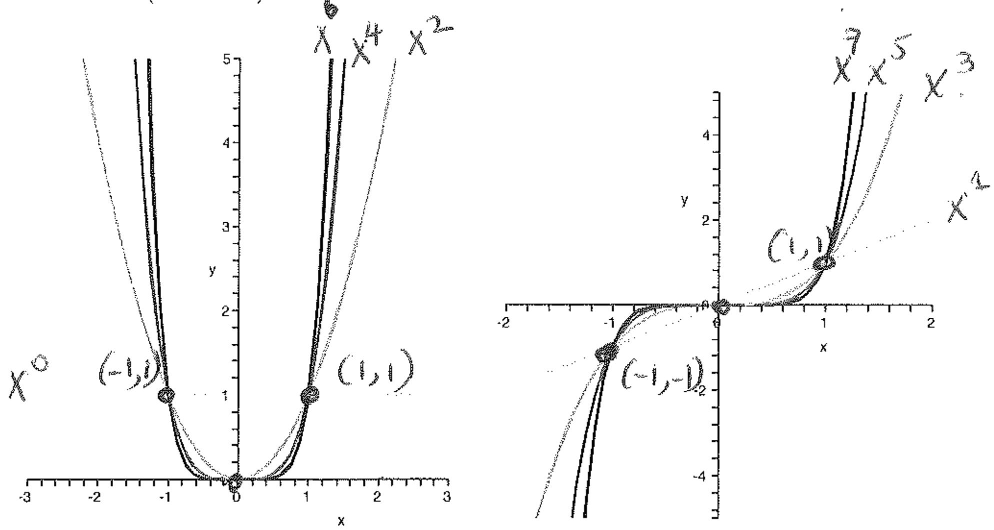


both $f(x)$ and $-x^3$.

In order to find out more about the overall graph of polynomial functions for large $|x|$, we therefore have to remind ourselves of the graphs of simple power functions of the kind x^n.

POWER FUNCTIONS OF THE KIND $f(x) = ax^n$ WITH n A NATURAL NUMBER

The shape of the graphs of functions of the kind $f(x) = x^n$ depends on whether n is an even or an odd number (see above).

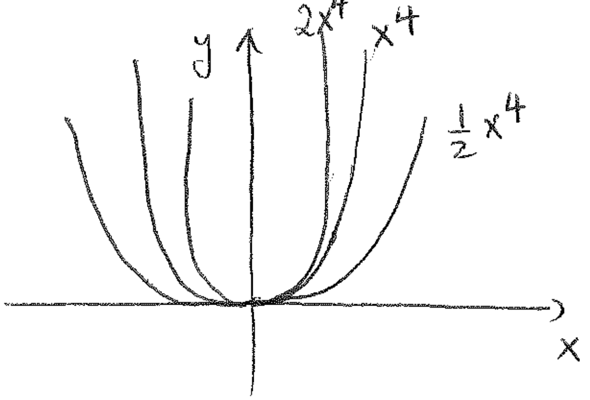


NOTE:

- Functions x^n where n is even "look like" x^2, and they are all even functions (symmetric about y-axis.
- Functions x^n where is odd "look like" x^3, they are all odd functions (with point symmetry about the origin).

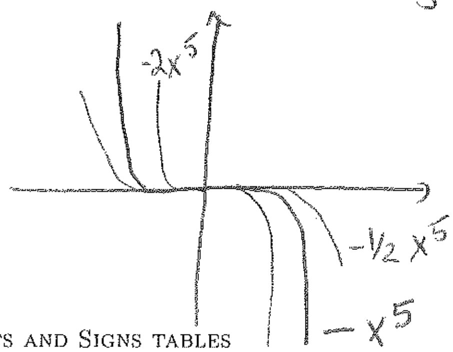
When the power is multiplied by a number a, note that

- If $a > 0$ then the shape remains the same but might be scaled



and similarly for odd functions

- If $a < 0$ then the shape is flipped over the x-axis (and might be scaled)

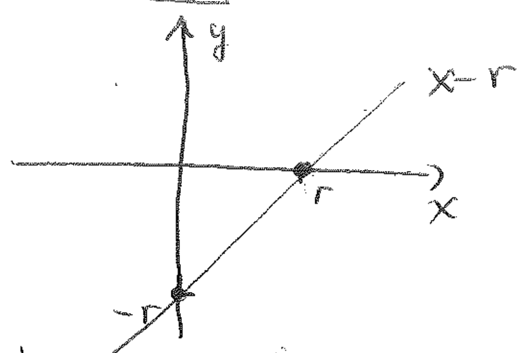


ROOTS AND SIGNS TABLES

So far we have used the expanded form to learn about the polynomial. We can also use the factored form to learn more about it, and graph the function with quite a lot of detail, but very little effort!

To do this, we first have to remember that the roots of the polynomial can be read directly from the factored form (see earlier). Then, we also have to remember that the basic factor $\boxed{x-r}$ (where r is one of the roots)

- $x - r = 0$ when $x = r$
- $x - r > 0$ when $x > r$
- $x - r < 0$ when $x < r$



Finally, we also have to remember that

- The product of 2 positive numbers is positive
- " " " negative numbers is positive
- " " one positive and one negative number is negative.

Using all of this, we can use a *Signs Table* to determine the sign, and therefore the overall shape, of any factored polynomial function $f(x)$.

IMPORTANT NOTE: Signs tables can only be used if the function is already broken down into its factors.

⚠ Make sure the polynomial is factored!

How to draw a signs table:

- Draw the table
- Write **all** the factors vertically on the left
- Write **all** the roots horizontally on the top (in the correct order)
- Draw vertical lines below each root
- Determine and write the sign of each factor; write zeros where appropriate.
- Multiply the signs in each interval to determine the sign of the function.

$x_1 \rightarrow$ roots go here, in order $\longrightarrow$

		x_1						
$(x-x_1)$	−	0	+	+	+	+

factors go here

completing the table

← tally up the signs

Examples of use of signs tables:

Example 1: Draw a signs table and sketch the function $f(x) = 4(x-1)(x+2)$.

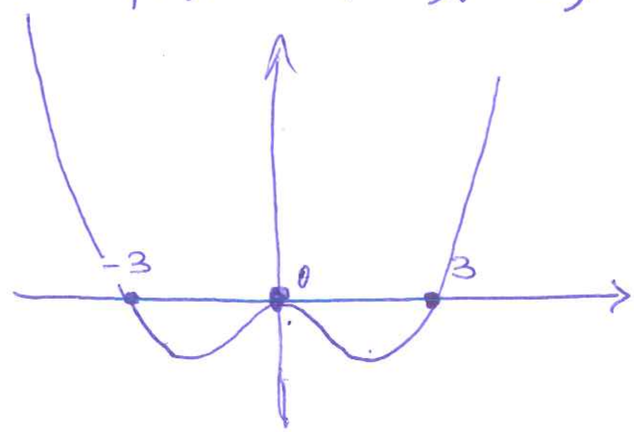
roots are 1, -2

		-2		1	
4	+		+		+
$x-1$	−		−	0	+
$x+2$	−	0	+		+
$f(x)$	+	0	−	0	+

Example 2: Draw a signs table and sketch the function $f(x) = x^2(x^2-9)$

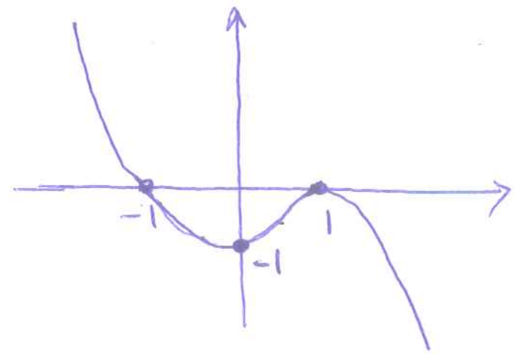
$f(x) = x^2(x-3)(x+3)$

		-3		0		3	
x	−		−	0	+		+
x	−		−	0	+		+
$x-3$	−		−		−	0	+
$x+3$	−	0	+		+		+
$f(x)$	+	0	−	0	−	0	+



EXAMPLE 3: Draw a signs table and sketch the function $f(x) = -(x+1)(x^2-2x+1)$ $= -(x+1)(x-1)^2$

		-1		1	
-1	$-$		$-$		$-$
$(x+1)$	$-$	0	$+$		$+$
$(x-1)^2$	$+$		$+$	0	$+$
	$+$	0	$-$	0	$-$



EXAMPLE 4: In which interval(s) is the function $f(x) = -(2-x)(x+3)^3$ positive?

$= (x-2)(x+3)^3$

	$+\infty$	-3		2	$+\infty$
(-1)	$-$		$-$		$-$
$2-x$	$+$		$+$	0	$-$
$x+3$	$-$	0	$+$		$+$
$x+3$	$-$	0	$+$		$+$
$x+3$	$-$	0	$+$		$+$
	$+$	0	$-$	0	$+$

Answer:

$(-\infty, -3] \cup [2, +\infty)$

or $\{ x \leq -3$ or $x \geq 2$

EXAMPLE 5: Find the domain of definition of $f(x) = \sqrt{x^5 - 2x^3 + 4x}$.

$\rightarrow$ we want to make sure that $x^5 - 2x^3 + 4x \geq 0$

$$x^5 - 2x^3 + 4x = x\underbrace{(x^4 - 2x^2 + 4)}$$

let $x^2 = u$

$u^2 - 2u + 4$

calculate

$D = (-2)^2 - 4(1)(4)$
$= 4 - 16 = -12$

$x^4 - 2x^2 + 4$ cannot be factored $\rightarrow$ either always > 0 or always < 0

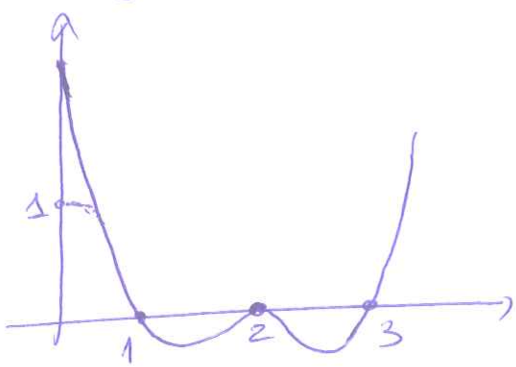
		0	
x	$-$		$+$
x^4-2x^2+4	$+$		$+$
x^5-2x^3+4x	$-$	0	$+$

$\Rightarrow \mathcal{D} = [0, +\infty)$

Let's now go back to our case study, and see if we can make a more educated guess as to the formula for $f(x)$ (or $f(x)$) given what we have just learned.

2.3.3 Case study: *How to build a roller-coaster*

To get the function to <u>not</u> cross the axis at $x=2$, lets try $(x-1)(x-2)^2(x-3)$, looks like



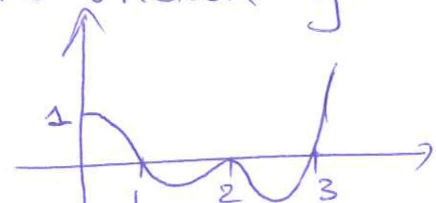
→ nearly right but

- does not flatten out at $x = 0$
- does not start at 1.

$h(x) = (x-1)(x-2)^2(x-3)$ is <u>not</u> even, but I can create an "even version of $h(x)$" by adding

$$h(x) + h(-x) = (x-1)(x-2)^2(x-3) + (-x-1)(-x-2)^2 \cdot (-x-3)$$

Then divide the entire function by 140 & voila ✓



In conclusion of this Chapter, note that polynomial come up in a vast number of applications in science:

- They are very simple functions, that often come up in real-life problems
- They can be used to model almost any data, at least for a limited range of the independent variable

In practice, finding the right polynomial to fit a particular dataset is rarely done by trial and error as we did – there are some very specific *fitting tools* that one can use to do it much more systematically. However, to use these tools well, it does help to know ahead of time for instance what order polynomial we will want to fit – and what we have learned today helps!