2.2 Quadratic functions

Textbook Sections 2.3, 2.4 and 2.6

2.2.1 Case study: *How to select the optimal pricing of Donelly's chocolates?*

Donnelly's in Santa Cruz (located by Bay and Mission) sell some of the finest chocolates in the US. This case study will show you how their company selects the pricing of their chocolates.

The cost of producing the chocolates is the following:

- *the rental of the facility costs \$200 per day,*
- *the salaries & benefits costs \$300/day*
- *the raw material for each chocolate costs \$1 per piece.*

What function $c(n)$ expresses the total production costs per day c as a function of the number of chocolates made n?

Total costs per day = Rental + salaries/benefits + raw material
= 200 + 300 + 1 · n
C(n) = 500 + n → a linear function

What should one bear in mind when setting the price of the chocolates?

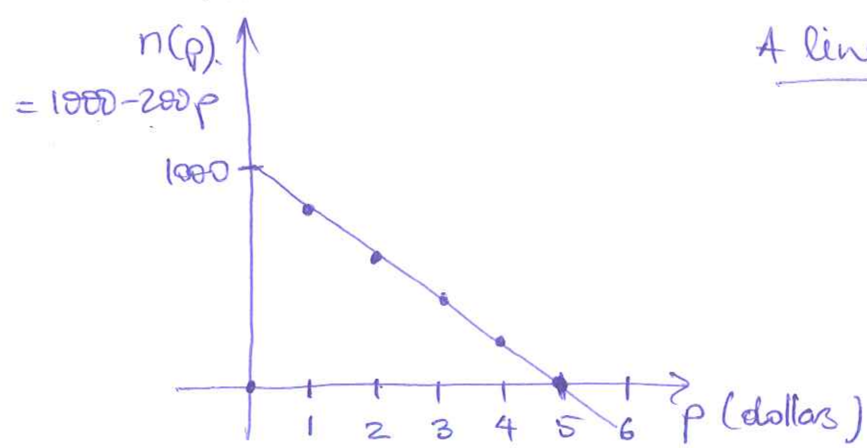
- Make profit ⇒ revenue must exceed costs!

BUT
- If sale price of chocolates is too high, not many will be sold (revenue will be too low)
- If sale price is too low, then revenue will also be too low.

somewhere in the middle, there is a sweet spot

To find the optimal pricing for the chocolates, the owners decide to conduct a market analysis, i.e. a poll of their customers, to determine how many chocolates anyone would buy depending on the price of the chocolates. After interviewing all of their customers over the course of a week, they estimate that the number of chocolates n they would be able to sell per day as a function of their price p would be $n(p) = 1000 - 200p$. Let's graph this function, and interpret its meaning:

A linear function



We now put all this information together, to find out how much money they would make each day m, as a function of the price of the chocolates p:

Net Money made per day = Total revenue − total costs

= number sold · price − costs

$$\text{Money}(p) = n(p) \cdot p - (500 + n(p))$$

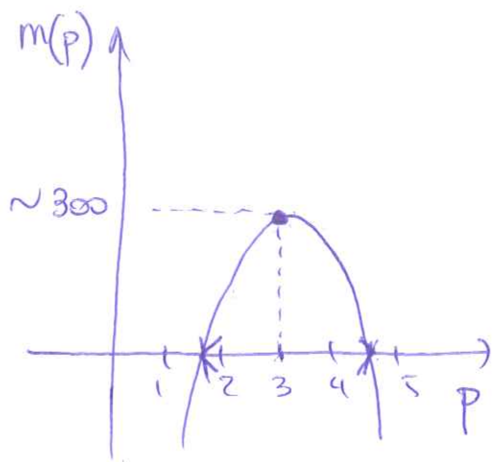
$$= n(p) \cdot p - 500 - n(p) = n(p)(p-1) - 500$$

$$= (1000 - 200p)(p-1) - 500$$

$$= 1000p - 1000 - 200p^2 + 200p - 500$$

$$\boxed{\text{money}(p) = -200p^2 + 1200p - 1500}$$

We can finally determine the optimal price for the chocolates, using Wolfram Alpha to graph the function $m(p)$.



→ There is indeed a sales price that maximizes profit

→ this price appears to be ~\$3/piece

→ the profit per day is close to \$300.

In this case study, we ended up solving our problem graphically. However, with the right mathematical tools, we can also do it without graphs. Let's learn more about quadratics.

2.2.2 Mathematical corner: Properties of quadratic functions

A quadratic function is a special type of polynomial function. The general expression for a quadratic function is

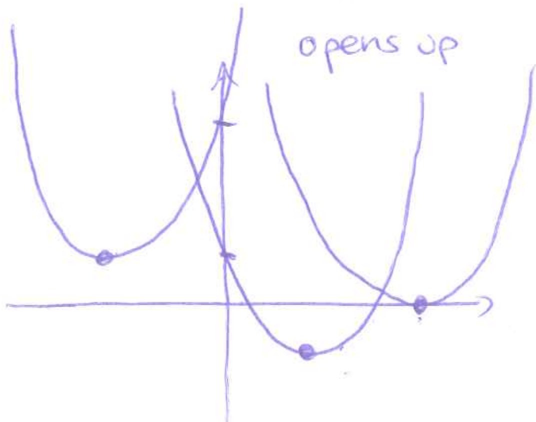
$$f(x) = ax^2 + bx + c \quad \text{where } a, b, c \text{ are real numbers.}$$

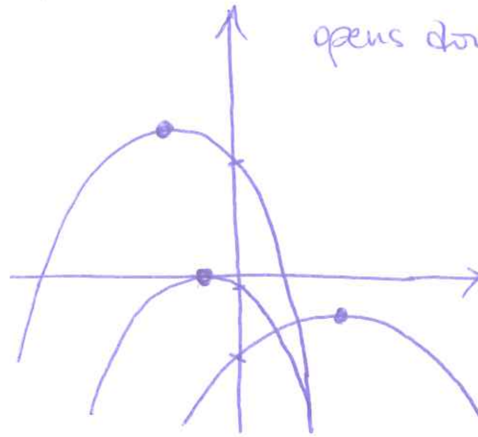
Example: $m(p) = -200p^2 + 1200p - 1500$

$$a = -200, \quad b = 1200, \quad c = -1500$$

The graph of all quadratic functions is called a *parabola*. The exact shape and position of the parabola depends on the coefficients of the quadratic. Different cases can arise:

- The parabola can open up or down
- The parabola can cross the x-axis twice, touch it once, or never touch it.





OPENING UP OR DOWN. Whether a parabola opens "up" or "down" can very easily be determined simply by inspection of the quadratic term ax^2 in the function.

Let's consider two examples of quadratic functions:

- $f(x) = 3x^2 - 2x - 1$
- $g(x) = -2x^2 + x + 1$

and graph them on Wolfram Alpha. We also compare their graphs with those of the functions $3x^2$ and $-2x^2$. We notice that:

- The graph of $f(x) = 3x^2 - 2x - 1$ looks like the graph of $y = 3x^2$ when x is large
- The graph of $g(x) = -2x^2 + x + 1$ looks like the graph of $y = -2x^2$ when x is large

This is in fact true of all quadratics!

- The graph of $f(x) = ax^2 + bx + c$ looks like the graph of $y = ax^2$ for large x
- This implies that the graph of $ax^2 + bx + c$ opens up if $a > 0$ opens down if $a < 0$

BEHAVIOR NEAR THE y-AXIS What the parabola looks like near the $y-$ axis (i.e. when x is close to 0) can *also* very easily be determined simply by inspection of the quadratic function, but this time, of the $bx+c$ bit.

Let's consider the functions $f(x)$ and $g(x)$ again, but this time zoom in the graph near $x = 0$. We also compare their graphs with those of the functions $-2x - 1$ and $x + 1$. We notice that:

- The function $f(x) = 3x^2 - 2x - 1$ looks like the line $y = -2x - 1$ for small x
- The function $g(x) = -2x^2 + x + 1$ looks like the line $y = x + 1$ for small x

Again, this is true for every quadratic!

- The graph of $f(x) = ax^2 + bx + c$ looks like the line $y = bx + c$ for small x
- This line is called the <u>tangent</u> to $f(x)$ at $x = 0$

VERTEX OF A PARABOLA AND VERTEX FORM

- The vertex of a parabola is the point at its minimum (if it opens up) or at its maximum (if it opens down)
- The coordinates of the vertex are $x_v = -\frac{b}{2a}$, $y_v = f\left(-\frac{b}{2a}\right)$
- Once the vertex (x_v, y_v) is known, we can put $f(x)$ in <u>vertex form</u> $f(x) = ax^2 + bx + c = \boxed{a(x - x_v)^2 + y_v}$

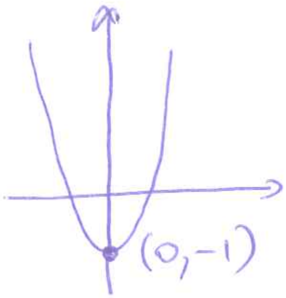
We already saw in the previous case study an example where we were interested in finding out what the coordinates of the vertex are (in that case, where the maximum of the graph was). We can now check analytically our graphical result. Indeed,

$$m(p) = -200p^2 + 1200p - 1500 \rightarrow a = -200,\ b = 1200,\ c = -1500$$

$$p_v = -\frac{1200}{2(-200)} = \frac{-1200}{-400} = 3 \checkmark$$

$$m(p_v) = -200p_v^2 + 1200p_v - 1500 = -200 \cdot (3)^2 + 1200 \cdot 3 - 1500 = 300 \checkmark$$

Here are further examples. In each case, let's use the vertex coordinates formula to find the position of the vertex, then use the vertex form of the quadratic to sketch the graph of the function using basic transformations.



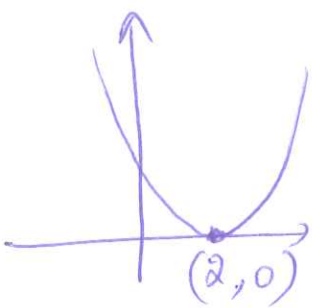
- $f(x) = x^2 - 1$ $a=1$ $b=0$ $c=-1$

$x_v = -\frac{b}{2a} = -\frac{0}{2\cdot 1} = 0$ $y_v = f(x_v) = x_v^2 - 1 = -1$

$f(x) = a(x-x_v)^2 + y_v = 1(x-0)^2 - 1 = x^2 - 1$

- $f(x) = -3x^2 + 4$ → already in vertex form $a(x-x_v)^2 + y_v$

$a=-3$ $x_v = 0$ $y_v = 4$



- $f(x) = x^2 - 4x + 4$ $a=1$ $b=-4$ $c=4$

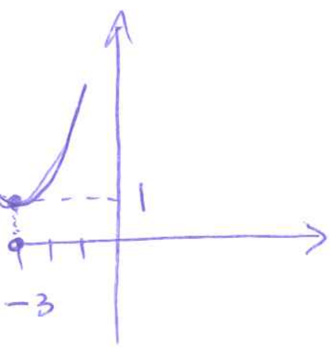
$x_v = -\frac{b}{2a} = -\frac{-4}{2\cdot 1} = 2$ $y_v = f(2) = 2^2 - 4\times 2 + 4 = 0$

$f(x) = a(x-x_v)^2 + y_v = 1\cdot(x-2)^2 = (x-2)^2$

- $f(x) = x^2 + 6x + 10$

$x_v = -\frac{b}{2a} = -\frac{6}{2\cdot 1} = -3$ $y_v = f(-3) = (-3)^2 + 6(-3) + 10 = 1$

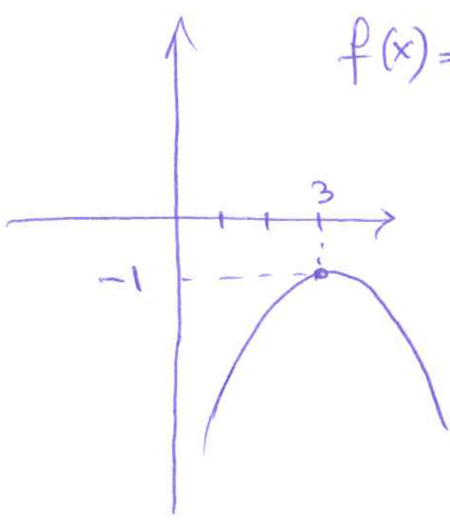
$f(x) = a(x-x_v)^2 + y_v = (x+3)^2 + 1$



- $f(x) = -5x^2 + 30x - 46$

$x_v = -\frac{b}{2a} = 3$ $y_v = -5(3)^2 + 30(3) - 46 = -1$

$f(x) = a(x-x_v)^2 + y_v = -5(x-3)^2 - 1$

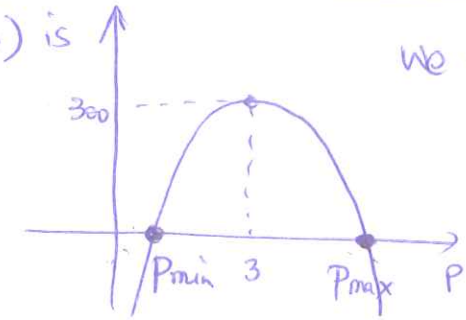


Once in vertex form $a(x-x_v)^2 + y_v$
the graph of $f(x)$ can be obtained by
- starting with ax^2
- move left/right to get $a(x-x_v)^2$
- move up/down to get $a(x-x_v)^2 + y_v$

2.2.3 Case study: *How to select the optimal pricing of Donnelly's chocolates?*

In the previous lecture, we learned how varying the selling price of Donnelly's chocolates affects the expected number of chocolates sold in any given day, and therefore the net profit that the company can make. We then selected the pricing to maximize profit. Alternatively, the company may decide to sell the chocolates a little below this price to attract new clients. How low a price can they sell their chocolates without losing any money? Conversely, how high a price could they try to sell them without losing money?

- We saw that $m(p)$ is



$\Rightarrow$ p_{min} is minimum possible price

$\Rightarrow$ p_{max} is maximum possible price

We see graphically $p_{min} \simeq \$1.8$

$p_{max} \simeq \$4.2$

- Mathematically, we can solve for p_{min} & p_{max} by solving $m(p) = 0 \Rightarrow -200p^2 + 1200p - 1500 = 0$

Let's put $m(p)$ in vertex form first:

$$m(p) = a(p-3)^2 + 300 = -200(p-3)^2 + 300$$

so $m(p) = 0$ becomes $-200(p-3)^2 + 300 = 0$

$$\Rightarrow -200(p-3)^2 = -300 \Rightarrow (p-3)^2 = \frac{-300}{-200} = \frac{3}{2}$$

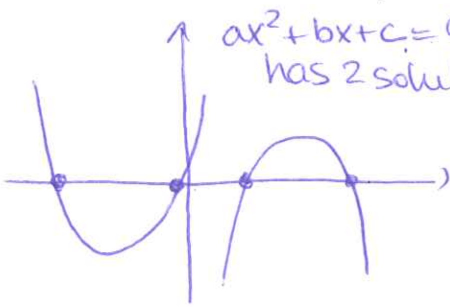
$$\Rightarrow p - 3 = \pm\sqrt{\frac{3}{2}} \Rightarrow \boxed{p = 3 \pm \sqrt{\frac{3}{2}}} = \begin{cases} 4.224\ldots \leftarrow p_{max} \\ 1.775\ldots \leftarrow p_{min} \end{cases}$$

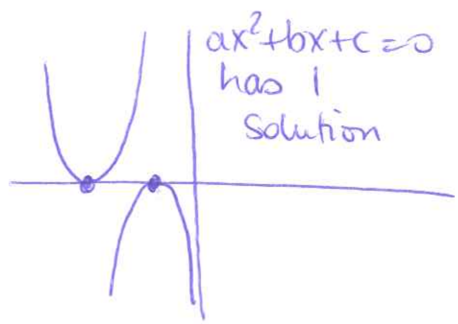
2.2.4 Mathematical corner: Roots of quadratics

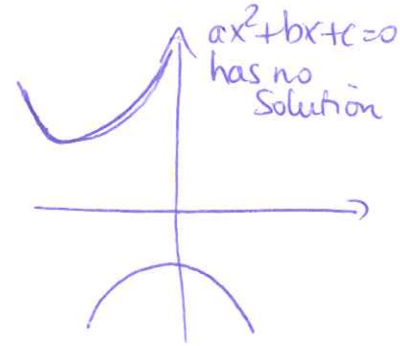
DEFINITION: The root of a quadratic (also called x-intercept) are the values of x where $f(x) = 0$

$\Rightarrow$ solutions of $\boxed{ax^2 + bx + c = 0}$

As we saw in the last lecture, some parabolas cross the x-axis, and some do not. Graphically, we know that there are 3 possible cases:







Since a parabola is the graph of a quadratic function, this means that some quadratic functions $f(x) = ax^2 + bx + c$ have 2 roots (i.e. 2 x-intercepts), some have one root (i.e. 1 x-intercepts), and some do not have any. In other words,

- $f(x) = ax^2 + bx + c$ can have 2 roots (2 x-intercepts)
- " " " " 1 root (1 x-intercept)
- " " can have 0 root (0 x-intercept)

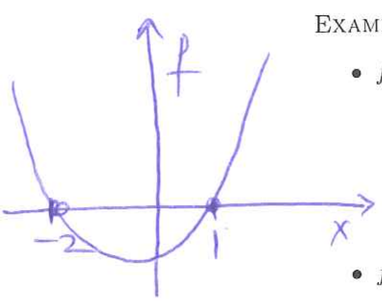
FACTORED FORM:

If the function $f(x)$ has two roots $\overset{\text{then}}{\underset{\text{then}}{\Longleftrightarrow}}$ it can be factored into the form $f(x) = a(x-x_1)(x-x_2)$ where x_1 & x_2 are the roots.

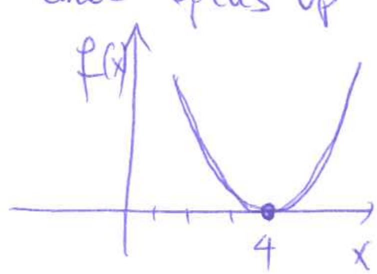
If $f(x)$ has one root x_1 $\overset{\text{then}}{\underset{\text{then}}{\Longleftrightarrow}}$ it can be factored as $f(x) = a(x-x_1)^2$

The symbol $\Leftrightarrow$ implies that there is a strict equivalence relationship between the two statements "$f(x)$ can be factored" and "$f(x)$ has roots": the first implies the second, and conversely, the second implies the first. In fact, it is quite easy to check that, in both cases, the second statement implies the first. Indeed, let us look for the solutions of $f(x) = 0$ from its factored form.

EXAMPLES



- $f(x) = 3(x-1)(x+2)$ $\Rightarrow$ $f(x) = 0$ has 2 solutions, $x = 1$ and $x = -2$, and opens up
- $f(x) = \frac{1}{2}(x-4)^2$ $\Rightarrow$ $f(x) = 0$ has 1 solution, $x = 4$ and opens up



The interesting thing about equivalence statements in logic is that if you have one, then you also have equivalence of the opposites:

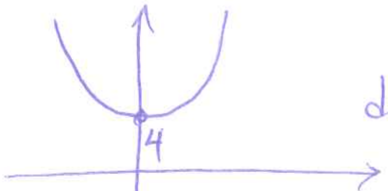
f can be factored $\Leftrightarrow$ f has roots $\quad$ f cannot be factored $\Leftrightarrow$ f has no roots

EXAMPLES

- $f(x) = x^2 + 4$ $\quad$ cannot be factored (sum of 2 squares)

$\Updownarrow$

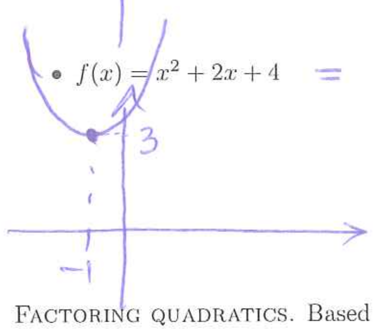
does not have roots



- $f(x) = x^2 + 2x + 4$ $= \underbrace{x^2 + 2x + 1} + 3 = (x+1)^2 + 3$

sum of 2 squares

cannot be factored.



FACTORING QUADRATICS. Based on what we just saw, it would be nice to have simple tricks to

- tell us when a quadratic has roots or not,
- or equivalently, factor the quadratic if it can be factored.

As it turns out, there are a few types of quadratics that can very easily be factored:

- $x^2 + 2ax + a^2 = (x+a)^2$
- $x^2 - 2ax + a^2 = (x-a)^2$
- $x^2 - a^2 = (x-a)(x+a)$

EXAMPLES

- $f(x) = x^2 - 2$ $= x^2 - (\sqrt{2})^2 = (x-\sqrt{2})(x+\sqrt{2})$

- $f(x) = 2x^2 - 3$ $= 2\left(x^2 - \frac{3}{2}\right) = 2\left(x^2 - \left(\sqrt{\frac{3}{2}}\right)^2\right)$

$= 2\left(x - \sqrt{\frac{3}{2}}\right)\left(x + \sqrt{\frac{3}{2}}\right)$

$= (\sqrt{2}x)^2 - (\sqrt{3})^2 = (\sqrt{2}x - \sqrt{3})(\sqrt{2}x + \sqrt{3})$

- $f(x) = x^2 + 6x + 9$ this is $x^2 + 2ax + a^2$ with $a = 3$

$\Rightarrow x^2+6x+9 = (x+3)^2$

- $f(x) = 2x^2 + 4\sqrt{5}x + 10$ $= 2(\underbrace{x^2+2\sqrt{5}x+5}_{\text{is } x^2+2ax+a^2 \text{ with } a=\sqrt{5}})$

$= 2(x+\sqrt{5})^2$

- $f(x) = -x^2 + 10x - 25$

$= -(x^2-10x+25) \leftarrow$ is $x^2-2ax+a^2$ with $a=5$

$= -(x-5)^2$

On the other hand, not every quadratic is in one of these three "ideal forms". What can we do if it isn't? As it turns out, another nice trick exists in that case, and is called "The quadratic formula".

THE QUADRATIC FORMULA. Given the quadratic $ax^2 + bx + c$,

- Calculate the discriminant $\boxed{D = b^2 - 4ac}$
- If $D < 0$ there are no solutions to the equation $ax^2 + bx + c = 0$, and the quadratic cannot be factored.
- If $D = 0$ there is one solution to the equation $ax^2 + bx + c = 0$, $\boxed{x_v = -\frac{b}{2a}}$ and the quadratic can be factored as

$$ax^2+bx+c = a\left(x+\frac{b}{2a}\right)^2 = a(x-x_v)^2$$

- If $D > 0$ there are two solutions to the equation $ax^2 + bx + c = 0$, which are $x_{1,2} = \frac{-b \pm \sqrt{D}}{2a}$

  and the quadratic can be factored as

$$ax^2+bx+c = a(x-x_1)(x-x_2)$$

NOTE: The vertex of a parabola is always half-way between the roots! Indeed,

let's calculate $\frac{x_1+x_2}{2} = \frac{\frac{-b+\sqrt{D}}{2a} + \frac{-b-\sqrt{D}}{2a}}{2}$

$-\frac{b}{2a} = \frac{\frac{-\cancel{2}b}{\cancel{2}a}}{2} = \frac{\frac{-b\cancel{+\sqrt{D}}+(-b)\cancel{-\sqrt{D}}}{2a}}{2}$

EXAMPLES:

$a=2 \quad b=-3 \quad c=1$

- What are the solutions (if any) to the equation $f(x) = 2x^2 - 3x + 1 = 0$? What is the factored form of f?

$$D = b^2 - 4ac = (-3)^2 - 4(2)(1) = 9-8 = 1 \rightarrow \text{two solutions}$$

$$x_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = \frac{-(-3) \pm \sqrt{1}}{2(2)} = \frac{3 \pm 1}{4} = \begin{cases} 1 \\ \frac{1}{2} \end{cases}$$

$$f(x) = 2x^2 - 3x + 1 = a(x-x_1)(x-x_2) = 2(x-1)\left(x-\frac{1}{2}\right)$$

- What are the solutions (if any) to the equation $f(x) = x^2 + x - 6 = 0$? What is the factored form of f?

$a=1 \quad b=1 \quad c=-6$

$$D = b^2 - 4ac = 1^2 - 4(1)(-6) = 1+24 = 25 \rightarrow 2 \text{ roots}$$

$$x_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = \frac{-1 \pm \sqrt{25}}{2(1)} = \frac{-1 \pm 5}{2} = \begin{cases} 2 \\ -3 \end{cases}$$

$$x^2 + x - 6 = a(x-x_1)(x-x_2) = (x-2)(x+3)$$

- What are the solutions (if any) to the equation $f(x) = -2x^2 - 8x - 8 = 0$? What is the factored form of f?

$a=-2 \quad b=-8 \quad c=-8$

$$D = b^2 - 4ac = (-8)^2 - 4(-2)(-8) = 64 - 64 = 0 \rightarrow 1 \text{ root}$$

$$x_v = -\frac{b}{2a} = -\frac{-8}{2(-2)} = -2 \Rightarrow f(x) = a(x-x_v)^2 = -2(x+2)^2$$

- What are the solutions (if any) to the equation $f(x) = -x^2 + x - 6 = 0$? What is the factored form of f?

$a=-1 \quad b=1 \quad c=-6$

$$D = b^2 - 4ac = (1)^2 - 4(-1)(-6) = 1 - 24 = -23$$

$\rightarrow$ no roots.

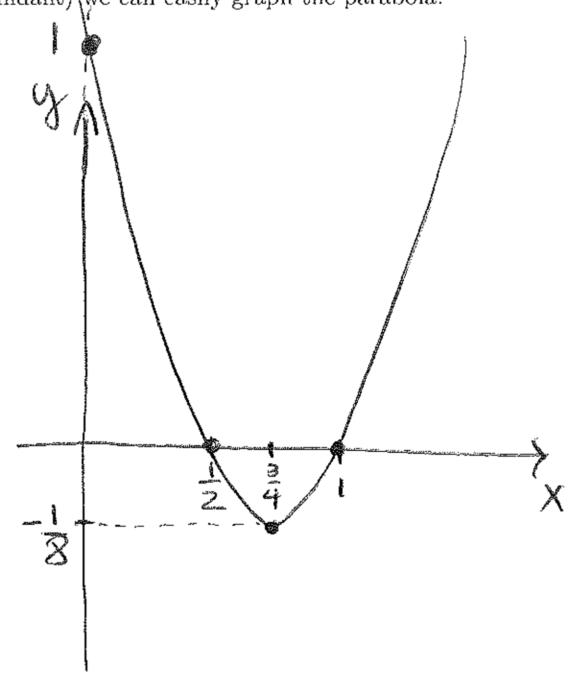
$\rightarrow$ can't be factored

Graphing quadratics. Let's now recap everything we know about the graphs of quadratic functions based on their mathematical expression. Given $f(x) = ax^2 + bx + c$,

- The graph is a parabola $(a \neq 0)$
- It opens up if $a > 0$, down if $a < 0$
- Its vertex is at $\left(-\frac{b}{2a}, \; a\left(-\frac{b}{2a}\right)^2 + b\left(-\frac{b}{2a}\right) + c\right)$
- The line $y = bx + c$ is tangent to the parabola at $x = 0$.
- Given $D = b^2 - 4ac$:
  - if $D < 0$ the graph does not cross the x-axis (no roots)
  - if $D = 0$ the graph just touches the x-axis at the vertex (1 root)
  - if $D > 0$ the graph crosses the x-axis twice at $x_{1,2} = \frac{-b \pm \sqrt{D}}{2a}$.

Based on all this information (much of which is actually redundant) we can easily graph the parabola.

Example 1: $f(x) = 2x^2 - 3x + 1$



- Opens up
- Vertex at $x_v = \frac{-(-3)}{2(2)} = \frac{3}{4}$

$$y_v = 2\left(\frac{3}{4}\right)^2 - 3\left(\frac{3}{4}\right) + 1$$
$$= -\frac{1}{8}$$

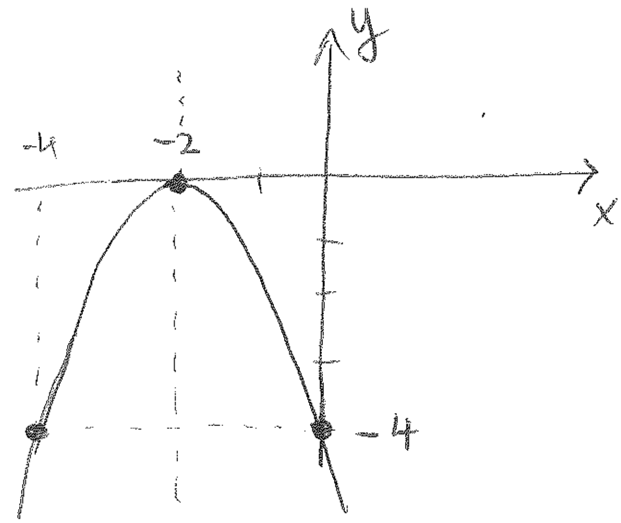
- Tangent line at $x=0$ is $y = -3x + 1$
- $D = 1$, $x_1 = 1$, $x_2 = \frac{1}{2}$ (see earlier).

EXAMPLE 2: $f(x) = -x^2 - 4x - 4$

- opens down
- vertex at $x_v = \frac{-(-4)}{2(-1)} = -2$

$$y_v = -(-2)^2 - 4(-2) - 4$$
$$= -4 + 8 - 4 = 0$$

$$D = (-4)^2 - 4(-1)(-4)$$
$$= 16 - 16 = 0 \checkmark$$



EXAMPLE 3: $f(x) = -x^2 + x - 1$

- opens down
- vertex at $x_v = \frac{-1}{2(-1)} = \frac{1}{2}$
- $y_v = -\left(\frac{1}{2}\right)^2 + \frac{1}{2} - 1$

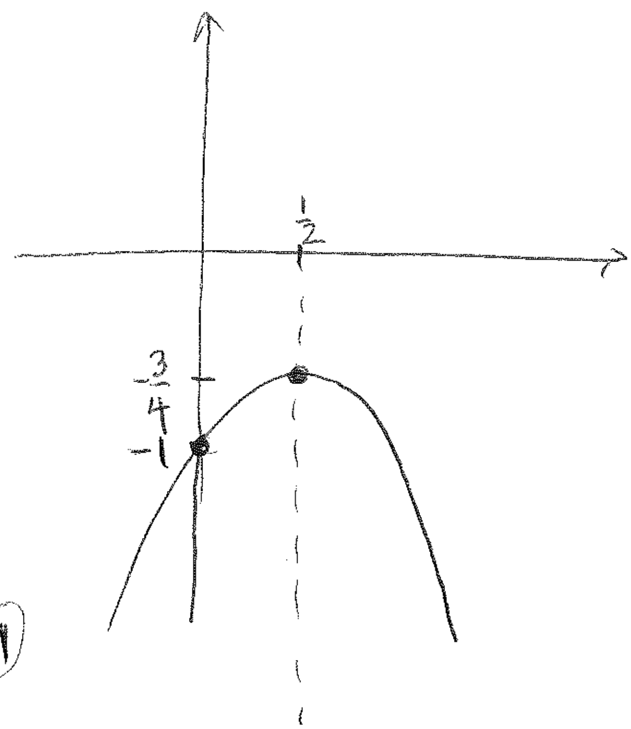
$$= -\frac{3}{4}$$

- $D = (1)^2 - 4(-1)(-1)$

$$= 1 - 4 = -3 < 0$$

no roots.

- Tangent ~~at~~ $x = 0$ $y = x - 1$



We can now also use this technique to solve the equation associated with our case study earlier, in a different way:

$$-200p^2 + 1200p - 1500 = 0 \quad ?$$

$$a = -200 \qquad b = 1200 \qquad c = -1500$$

$$D = (1200)^2 - 4(-200)(-1500) = 240000$$

$$p_{1,2} = \frac{-1200 \pm \sqrt{240000}}{2(-200)} = \frac{-1200 \pm \sqrt{24}\overbrace{\sqrt{10000}}^{100}}{-400}$$

$$= \frac{-12 \pm \sqrt{24}}{-4} = 3 \pm \sqrt{\frac{3}{2}} \quad \text{because} \quad \sqrt{\frac{24}{16}} = \sqrt{\frac{3 \times \cancel{8}}{2 \times \cancel{8}}}$$

Finally, it is worth noting that this method can also help solve a few higher-order equations that can be reduced to a quadratic, as in these examples:

- What are the solutions (if any) to the equation $f(x) = x^6 - 3x^3 - 9 = 0$?

If use intermediate variable $u = x^3$, then

$u^2 = x^6$ so $x^6 - 3x^3 - 9 = 0$

$\Leftrightarrow \quad u^2 - 3u - 9 = 0$

$D = (-3)^2 - 4(1)(-9) = 9 + 36 = 45 > 0 \rightarrow$ 2 solutions

$$u_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = \frac{3 \pm \sqrt{45}}{2} \longrightarrow x_{1,2} = \left(\frac{3+\sqrt{45}}{2}\right)^{\frac{1}{3}}, \left(\frac{3-\sqrt{45}}{2}\right)^{\frac{1}{3}}$$

- What are the solutions (if any) to the equation $f(x) = x^4 - 2x^2 - 3 = 0$?

$\rightarrow$ use intermediate variable $u = x^2$

$u^2 = x^4$

$\rightarrow \quad u^2 - 2u - 3 = 0$

$\rightarrow \quad D = (-2)^2 - 4(1)(-3) = 4 + 12 = 16$

$$\rightarrow \quad u_{1,2} = \frac{-(-2) \pm \sqrt{16}}{2(1)} = \frac{2 \pm 4}{2} = \begin{cases} 3 \\ -1 \end{cases}$$

$$\Rightarrow \begin{cases} x^2 = 3 \rightarrow x = \pm\sqrt{3} \\ x^2 = -1 \longrightarrow \text{no solution} \end{cases}$$

$\rightarrow$ So solutions to $x^4 - 2x^2 - 3 = 0$ are $x = -\sqrt{3}$ and $x = \sqrt{3}$