Chapter 2

Polynomial functions

In this Chapter we will study a large class of functions called "polynomial functions", which all take the form:

The simplest example of these polynomial functions are the linear functions.

2.1 Linear functions

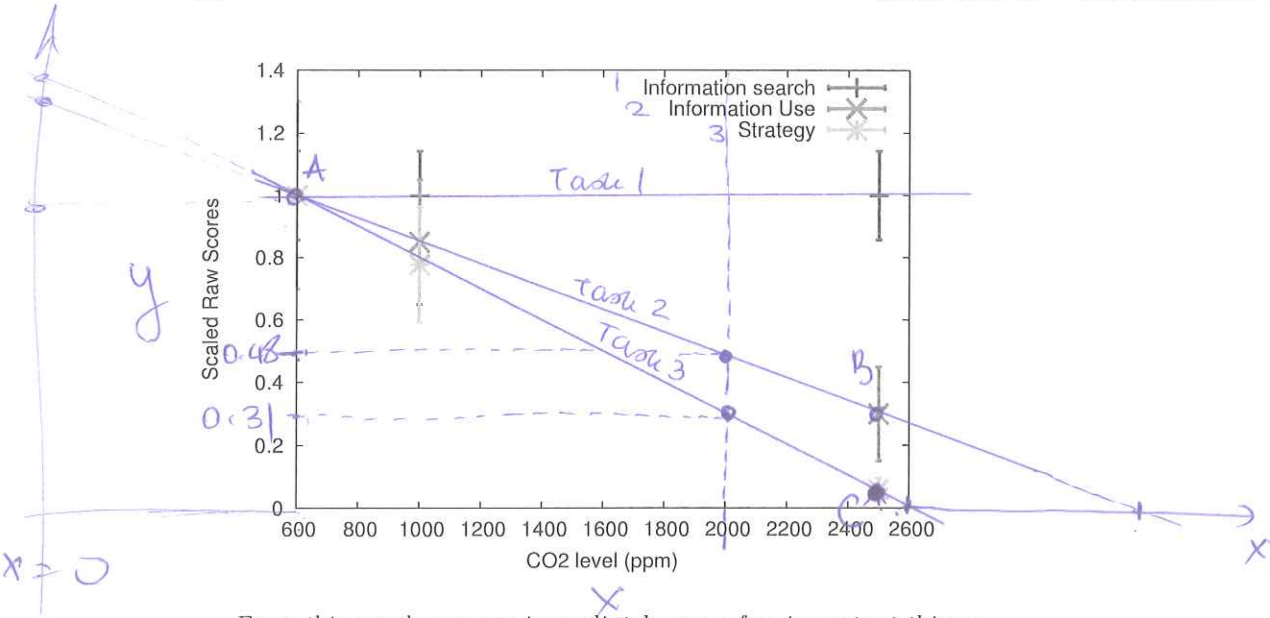
Textbook Section 2.1 and 2.2.

2.1.1 Case study: CO_2 as an indoor pollutant

This case study is based on the findings of a paper entitled "Is CO_2 an indoor pollutant? Direct effects of low to moderate CO_2 concentrations on human decision-making performance.", by a team of researchers from Upstate Medical University (Syracuse, NY) and from Lawrence Berkeley National Labs, in 2012 (see website for a link). Researchers asked 22 volunteers to take part in computerized decision-making tests that each take 2.5 hours, while being in an environment containing more CO_2 than normal, at levels of 600 ppm (considered normal for indoor environments), 1000ppm (slightly above normal), and 2500ppm (significantly above normal). None of these levels are thought to have long term harmful effects to health, but they were demonstrated in this experiment to have statistically significant impacts on various aspects of the decision making processes. Nine kinds of tasks were tested, ranging from basic activities, to information search and usage, to basic use of strategy, using a well-known and well calibrated decision-making test. The following table summarizes test results for 3 of these tasks, scaled to their "normal value" at a CO_2 concentration of 600ppm (so by construction, the scaled results is equal to one in that case).

CO_2	Information Search	Information Use	Strategy
600 ppm	1 ± 0.15	1 ± 0.3	1 ± 0.18
1000 ppm	1 ± 0.15	0.85 ± 0.2	0.77 ± 0.18
2500 ppm	1 ± 0.15	0.3 ± 0.15	0.06 ± 0.04

The errors are related to the spread in the test scores of each individual participants, while the main value reported is the average of all the test scores. The same information is presented in the following graph:



From this graph, we can immediately see a few important things:

- For a given task, the data lies in straight line
- For Task 1 (Info search), the score is independent of CO_2 level
- For Task 2 & 3 (Info use & Strategy), the scores drop with increasing CO_2 level, and the effect is more pronounced for the more complex task.

However, can we be more quantitative about some of these statements? And can we use this information to predict, what the relative decision-making performance of someone may be as a function of the ambient CO_2 concentration for any possible input values of the latter? Or, above which level the performance on a given task becomes so poor as to be useless? To do this, we need to create a function that takes as input the value of the ambient CO_2 concentration, and returns the scaled raw score. With such knowledge, one would be able to better predict the effects of CO_2 on the performance of workers in a company, or of students in test-taking environments, depending on the tasks they have to perform. Let us now learn a little bit more about linear functions, which we shall see are quite relevant for this case study.

2.1.2 Mathematical corner: Definition and basic properties of linear functions

DEFINITION: A linear function can be written as $f(x) = mx + b$ where m & b are real numbers.

Example: $f(x) = x + 1$ $\quad$ $f(x) = -3x - 4$

$g(y) = 5y + 2$ $\quad$ $g(y) = 10$

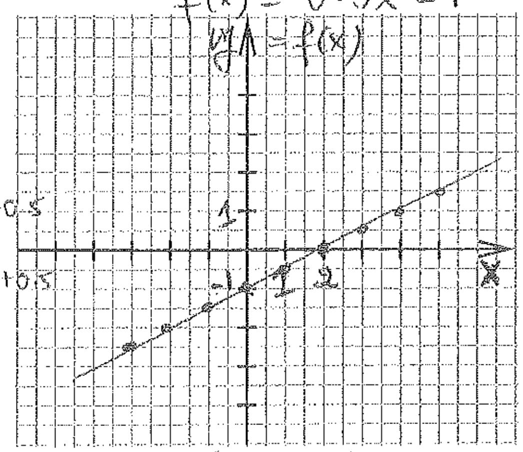
$P_{store}(n) = 16$ $\quad$ $P_{online}(n) = 10 + 14n$

DOMAIN OF DEFINITION OF A LINEAR FUNCTION: A linear function is defined for all real numbers, $\mathcal{D} = \mathbb{R}$

EXAMPLES OF LINEAR FUNCTIONS AND THEIR GRAPHS:

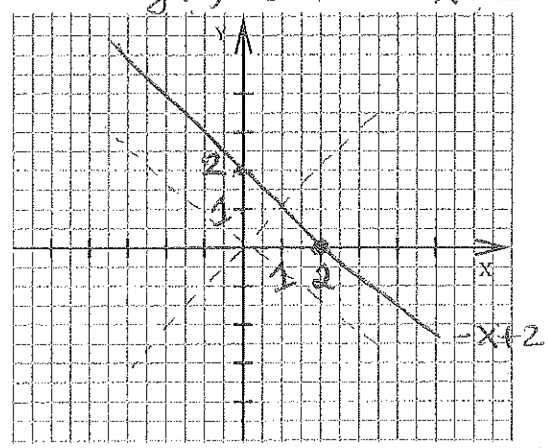
x	$0.5x-1$
-3	-2.5
-2	-2
-1	-1.5
0	-1
1	-0.5
2	0
3	0.5

$f(x) = 0.5x - 1$



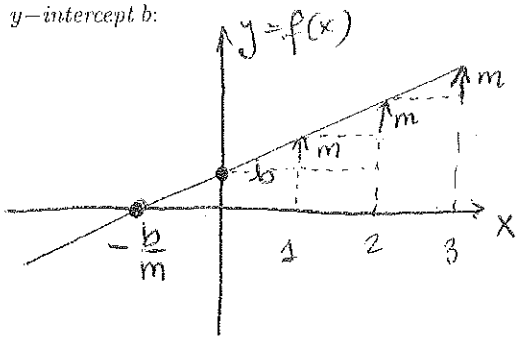
A straight line with slope 0.5 (0.5 up for each 1 to the right)

$g(x) = 2 - x = -x + 2$



A straight line with slope -1 (1 down for each 1 to the right)

GRAPH OF A LINEAR FUNCTION: The graph of $y = f(x) = mx + b$ is a straight line with *slope* m and $y-$*intercept* b:



The y-intercept is where the graph $y = f(x)$ intersects the y-axis. For a linear function $f(0) = m \cdot 0 + b = b$
$\rightarrow$ y intercept is $(0, b)$

The slope is defined as the change in y divided by change in x. If the change in x is 1 then

$$\text{slope} = \frac{f(x+1) - f(x)}{1} = \frac{[m(x+1)+b] - [mx+b]}{1}$$
$$= \cancel{mx} + m + \cancel{b} - \cancel{mx} - \cancel{b} = m$$

Note that we can also define the $x-$*intercept*:

The x-intercept is where the graph of $y = f(x)$ intersects the x-axis. (ie, when $y = 0$). To find it we solve the equation $f(x) = 0$ for x:

$$mx + b = 0 \longrightarrow mx = -b \rightarrow \boxed{x = -\frac{b}{m}}$$

EXAMPLES:

- $f(x) = 0.5x - 1$ y-intercept is -1 slope $= \frac{1}{2}$
  x-intercept is: $f(x)=0 \Rightarrow 0.5x-1=0 \Rightarrow x=\frac{1}{0.5}=2$
- $f(x) = 2-x$ y-intercept is 2 slope is -1
  x-intercept is: $f(x)=0 \Rightarrow 2-x=0 \Rightarrow x=2$
- $f(x) = -\frac{1}{2}x+4$ y-intercept is 4 slope $= -\frac{1}{2}$
  x-intercept: $f(x)=0 \Rightarrow -\frac{1}{2}x+4=0 \Rightarrow x=8$

BASIC PROPERTIES OF A LINEAR FUNCTION:

- A linear function is continuous
- It is increasing for all x if $m>0$
- " " decreasing for all x if $m<0$
- It is constant if $m=0$ (then $f(x)=b$)

2.1.3 Case study: CO_2 as an indoor pollutant (part 2)

Going back to our case study, we want to find a function that takes as input the CO_2 level, and returns, for any input value, the predicted relative test score for each of the three tasks.

Since the points on the graph lie in a straight line, we can infer that this function is likely a linear function $f(x)=mx+b$ (m, b are different for each task).

Let us look at Task 1 (Information search) first.

Since the points all have $y=f(x)=1 \Rightarrow$ we see that the function is the constant function $\boxed{f(x)=1}$

For Tasks 2 (Information use) and Task 3 (Strategy), it's a little more complicated, but not much.

On the one hand we want $f(x)=mx+b$ (linear)
" " other hand we want it to fit the available data so $y=f(x)$ for each point (x,y)

$\Rightarrow$ we need to find, for each task, m and b such that $y=mx+b$ for each point

$\Rightarrow$ This is equivalent to finding the equation of a line that goes through some points.

As we learned in Algebra, this can be done using the point-slope formula:

- find the slope through 2 points A & B: $m = \frac{y_B - y_A}{x_B - x_A}$
- Use $y = m(x - x_A) + y_A$

For task 2 (information use), we have $A(600, 1)$ $B(2500, 0.3)$

$$m = \frac{0.3 - 1}{2500 - 600} = -\frac{0.7}{1900} \cong -0.00037$$

so $y = f(x) = -0.00037(x - 600) + 1$ $= 1.222 - 0.00037x$

For task 3 (strategy), we have $A(600, 1)$ $C(2500, 0.06)$

$$m = \frac{0.06 - 1}{2500 - 600} = \frac{-0.94}{1900} \cong -0.00049$$

so $y = f(x) = -0.00049(x - 600) + 1$ $= 1.294 - 0.00049x$

To conclude, now that we know m and b for each task, we can finally write that the function describing the relationship between relative test performance and CO_2 concentration is

Task 1 (Info search): $f(x) = 1$

Task 2 (Info use): $f(x) = -0.00037(x - 600) + 1$

Task 3 (Strategy): $f(x) = -0.00049(x - 600) + 1$

Let's now interpret these mathematical formulae. We see that the slope becomes more and more negative as the complexity of the thought processes required to complete the task become:

$$m_1 = 0 > m_2 = -0.00037 > m_3 = -0.00049$$

→ CO_2 has a more negative effect on complex thought processes than simple ones.

This confirms our qualitative inspection of the figure, but gives us a way to **quantify** *the effect. We can also use this information in a predictive way, to estimate what the decrease in test performance (relative to the standard test at 600ppm) would be for any input concentration of CO_2. For instance, at 2000 ppm, the relative performance in each task would be*

Task 1: $f(2000) = 1$

Task 2: $f(2000) = -0.00037(2000 - 600) + 1$
$= 0.482$ (only 48% correct answers relative to normal CO_2 level)

Task 3: $f(2000) = -0.00049(2000 - 600) + 1 = 0.314$ ⇒ 31% correct answers.

Similarly, any relative test score close to 0 or even below 0 indicates systematically dysfunctional decision making. Above which CO_2 concentration does this happen? Again, we can answer this equation either by looking at the graph, or mathematically.

Mathematically, we want to find when $f(x)=0$

$\rightarrow$ Task 1: $f(x)=0 \Rightarrow 1=0 \Rightarrow$ never

$\rightarrow$ Task 2: $f(x)=0 \Rightarrow -0.00037(x-600)+1=0$

$\Rightarrow x-600 = \frac{1}{0.00037} \Rightarrow x \simeq 3300$

Task 3: $f(x)=0 \Rightarrow -0.00049(x-600)+1=0$

$\Rightarrow x-600 = \frac{1}{0.00049} \Rightarrow x \simeq 2640$

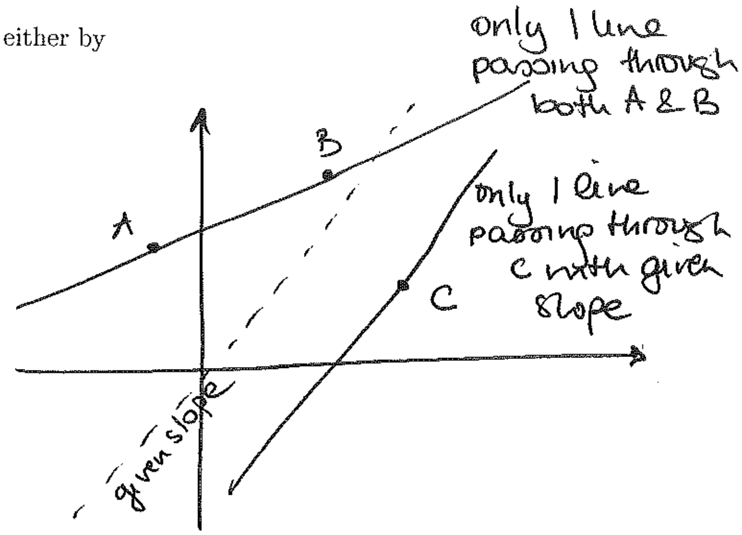
These results all strongly suggest that offices where research is done, or where non-trivial managerial decisions are made, should be well-ventilated to avoid the negative effects of CO_2.

2.1.4 Mathematical corner: Properties of lines (a little bit of geometry)

Studying linear functions is easier if we remember a little bit about properties of lines, since the graph of a linear function *is* a line.

Geometrically speaking, a line is *uniquely* defined either by

- Two points
- A point and a slope



CONSEQUENCE: The equation of a line can be found either given 2 points on the line OR give one point & one slope.

Note that if you know the coordinates of the two points, you can calculate the slope of the line going through the points:

$$m = \frac{y_A - y_B}{x_A - x_B} = \frac{y_B - y_A}{x_B - x_A} = \frac{\text{change in } y}{\text{change in } x}$$

Also, remember that

- Two lines with the same slope are parallel
- A line parallel to the x-axis has slope 0 (so its equation is $y=b$)
- A line parallel to the y-axis has equation $x=c$

(but is not the graph of a function)

Once the slope of a line is known, there are two ways of writing the line equation:

- The slope-intercept formula (if you know the y−intercept): $y = mx+b$
- The point-slope formula (if you know a point on the line): $y = m(x-x_A)+y_A$

or $y - y_A = m(x - x_A)$

EXAMPLES:

- Finding a line going through one point, with a "given" slope:

Example slope is 1, point is A (5,2)

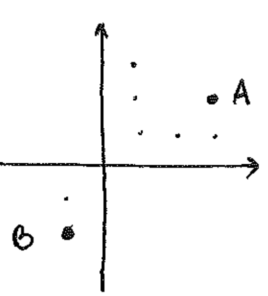
$\rightarrow \quad y = 1(x-5)+2 = x-5+2 = x-3$

Line parallel to $y=-3x+2$, going through (1,1)

$\rightarrow \quad y = -3(x-1)+1 = -3x+3+1 = -3x+4$

- Finding a line going between two points:

Example: $A(\overset{x_A}{3},\overset{y_A}{2}) \quad B(\overset{x_B}{-1},\overset{y_B}{-2})$



Slope: $m = \dfrac{-2-2}{-1-3} = \dfrac{-4}{-4} = 1$

$\Rightarrow \quad y = 1(x-3)+2 = x-3+2 = x-1$

Check: A(3,2): $y = 2 = x-1 = 3-1 = 2$ ✓

B(−1,−2): $y = -2 = x-1 = -1-1 = -2$ ✓