Chapter 1

The notion of functions

Textbook Chapter 1

1.1 The concept of functions

Although the concept of functions was invented a very long time ago, it is very easy today to gain an intuitive notion of what functions are because of their natural role in most computer and/or web-based applications, in engineering, and in economics, etc. Let us study a very simple example of functions that arise in every day life.

1.1.1 Case Study: *Online vs. in-store purchase of bulk items*

When buying bulk items online, the price is often somewhat cheaper than in store, but one usually needs to add an extra fee for shipping. One may therefore wonder what the cheapest option is, depending on the quantity we would like to purchase. Consider for instance buying diapers:

- *In store, a pack of 30 diapers usually sells for about $16.*
- *Online, the same pack usually sells for $14, but there is a $10 shipping fee to add on the total order.*

A very simple way of finding out which option is the cheapest is to make a table to compare the price one would have to pay to buy n packs:

n	1	2	3	4	5	6	7	Formula
In store	16	32	48	64	80	96	112	$16n$
Online	24	38	52	66	80	94	108	$10+14n$

We see from the table that buying online becomes cost-effective when: $n > 5$

Note that neither the cashier in the store, nor the online retailer, would calculate the price this way. Instead, their computer would calculate the price automatically for a given n simply by applying a simple formula in each case:

- Price online is a function of n: $P_{online}(n) = 10 + 14n$
- " in store " " n: $P_{store}(n) = 16n$

These formulae are simple **mathematical rules**, *that give the price one has to pay* **as a function** *of the number of diapers, in each case. These kinds of mathematical formulas commonly arise in any computerized pricing system, but also much more generally in a vast number of applications, that we will gradually explore this quarter. They are called* **functions**. **See slides for more examples.**

1.1.2 Mathematical corner: Functions defined by a mathematical rule

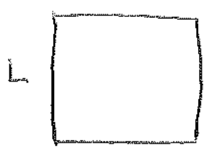
While the concept of functions is very general, we will focus in this course only on functions that are defined by a mathematical rule, as in the simple examples we just saw. Let's now define functions more rigorously, and learn a little vocabulary.

DEFINITION: A function defined by a mathematical rule takes one number, applies a formula, and returns another number, We usually write it as

$$x \longrightarrow f(x) = \text{the formula applied to } x$$

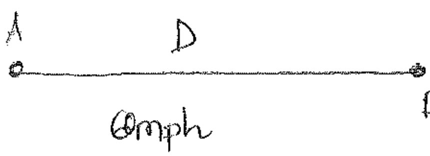
EXAMPLES THAT ARISE FROM REAL WORLD PROBLEMS:

- What is the function that returns the area of a square A given the length of its side L?



$$A = f(L) = L^2$$

- What is the function that expresses the time T that it takes to travel somewhere at 60mph, given the distance D?



$$T = f(D) = \frac{D}{60}$$

EXAMPLES THAT ARISE FROM A MATHEMATICAL EXPRESSION: More generally, we can define nearly any function using standard mathematical operations, as in the examples below.

- $f(x) = \sqrt{a + i + x^2}$ → takes x, squares it, add a and i then square roots result.
- $g(x) = |x|$ → takes x and returns the positive part
- $v(y) = y^2$ → takes y and squares it
- $f(x) = e^x$ → takes x, and exponentiate it

NOTE:

The name of the function doesn't have to be f

The name of the variable doesn't have to be x.

DEFINITIONS: If a quantity y is related to a quantity x via the formula $y = f(x)$ then we can say that

- x is the independent variable (input variable)
- y is the dependent variable (output variable)

EXAMPLES:

- $y = g(x) = |x|$ $\begin{cases} \text{indep var is } x \\ \text{dep var is } y \end{cases}$
- $z = h(y) = \sqrt{y+1}$ $\begin{cases} \text{indep var is } y \\ \text{dep var is } z \end{cases}$
- $a = d(b) = b + c$ $\begin{cases} \text{indep var is } b \\ \text{dep var is } a \end{cases}$

To evaluate a mathematical function for any input value of the independent variable, simply replace it in the mathematical formula by the desired number.

EXAMPLES FROM MATHEMATICS:

$$g(x) = |x| \longrightarrow g(-\pi) = |-\pi| = \pi$$

$$h(y) = \sqrt{y+1} \longrightarrow h(8) = \sqrt{8+1} = \sqrt{9} = 3$$

$$d(b) = b + c \longrightarrow d(3) = 3 + c \qquad d(c) = c + c = 2c$$

EXAMPLES FROM REAL LIFE:

- To find the cost of 10 diaper bags online and in-store, we calculate:

$P_{online}(10) = 10 + 14 \times 10 = 150$ $\qquad$ $P_{store}(10) = 16 \times 10 = 160$

- To find the area of a square of length 2, we calculate: $L=2$ $\quad A = f(2) = 2^2 = 4$

A function can also be applied to other variables and even whole expressions instead of numbers. In that case, simply replace the dependent variable by the input expression. Although this may seem strange at first, we will see later why this can be very handy.

EXAMPLES:

- $g(x) = |x| \longrightarrow g((a^2+1)) = |a^2+1|$
- $h(y) = \sqrt{y+1} \longrightarrow h(\square) = \sqrt{\square + 1}$
- $d(b) = b + c \longrightarrow d(c^2 - c + 1) = \underbrace{c^2 - \cancel{c} + 1}_{b} + \cancel{c} = c^2 + 1$

DOMAIN OF DEFINITION OF A FUNCTION: The *Domain of Definition* of a function f consists of all of the values x for which we **are allowed to** or **want to** assign a value $y = f(x)$.

- **"allowed to"** refers to the mathematical rules, i.e. when are you allowed to apply that rule to x

EXAMPLES:

1. $f(x) = \frac{1}{x-1}$ Not allowed division by 0 $\rightarrow$ $x - 1 \neq 0$

$\Rightarrow$ $(x \neq 1)$ $\Rightarrow$ $\mathcal{D} = \{x \mid x \neq 1\} = (-\infty, 1) \cup (1, +\infty)$

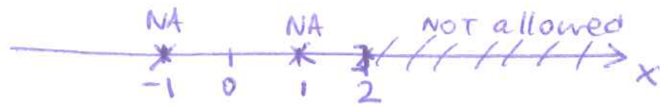
2. $f(x) = \sqrt{x-1}$

Not allowed $x - 1 < 0$ $\Rightarrow$ Not allowed $x < 1$

$\Rightarrow$ x must be greater than 1 $\mathcal{D} = [1, +\infty)$

3. $f(x) = \frac{\sqrt{2-x}}{x^2-1}$ Need $x^2 \neq 1$ and $2 - x \geq 0$

$\rightarrow$ Need $x \neq 1$ or $x \neq -1$ and $2 \geq x$



$\rightarrow$ $\mathcal{D} = (-\infty, -1) \cup (-1, 1) \cup (1, 2]$

- **"want to"** refers to the physical problem considered, i.e. what are the values of x that make sense?

EXAMPLE : What is the domain of definition of the price of diapers' bags)

$\rightarrow$ I can only buy, 1, 2, 3, 4 .... (a positive, integer number of diaper bags)

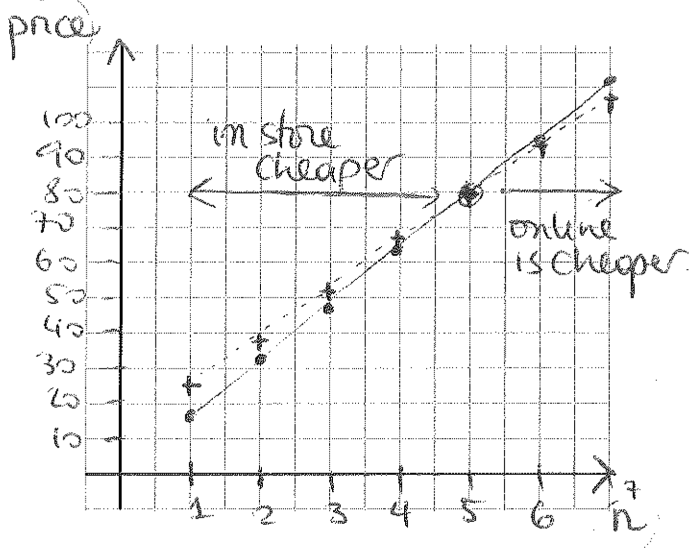
$\Rightarrow$ $\mathcal{D} = \mathbb{N} - \{0\}$

1.2 Functions and graphs

Merely looking at tables of numbers, or mathematical expressions, is not always particularly useful. However, **graphing** functions can often provide simple and useful answers to questions we may have. Let's go back to the case study to see how this works.

1.2.1 Case Study: *Online vs. in-store purchase of bulk items (part 2)*

Let us go back to the tables of numbers created earlier, and **graph** *of the price of the purchase vs. the number of packs bought. We get:*



∘ in store purchase
+ online purchase

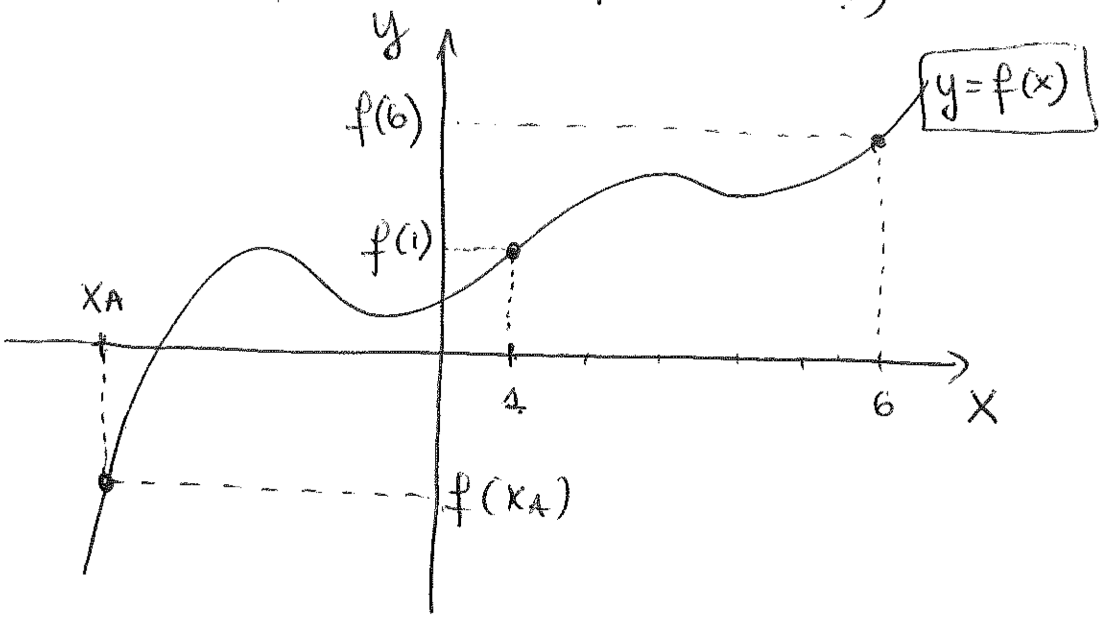
- If $n < 5$ then price in store is lower (line is lower) than online
- If $n > 5$ then price online is lower (line is lower) than in store

So we have found two different ways of answering the question of **which is cheaper : buying online or buying in the store?** *One is mathematical (applying the formula/function to calculate both prices, and comparing the values to see which is the smaller one) and one is graphical (plotting the price vs. the number of diapers bought and looking at which lies below).* **See slides for more examples.**

As it turns out, **functions and their corresponding graphs carry exactly the same information**, so the graphs can be used as a simple **visual representation** of the mathematical expression, which are arguably much more easily analyzed and understood. In what follows, we learn a little more about functions and their graphs, and what can be inferred from one about the other.

1.2.2 Mathematical corner: Graphs of standard functions and graphing techniques

DEFINITION: The graph of a function is the set of all the points (x, y) such that $y = f(x)$

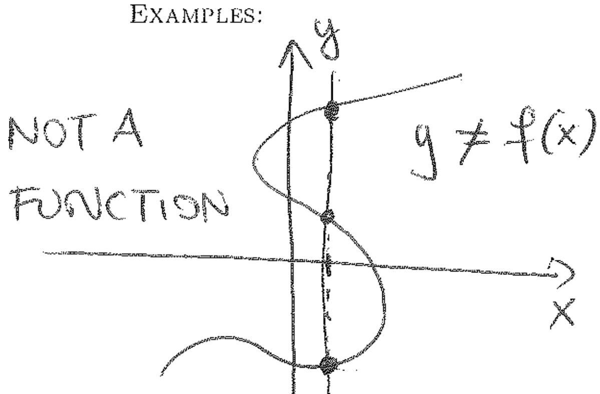


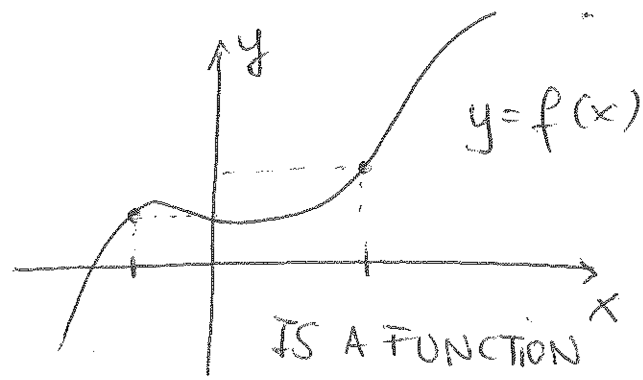
SINGLE VALUE PROPERTY:

- Saying "$y = f(x)$" only makes sense when there is a single value of y corresponding to each value of x. This means that while every function has a graph, not every graph can be the graph of a function!

- THE VERTICAL LINE TEST: A *graph* corresponds to a *function* only if it passes the *Vertical Line Test*: if any vertical line on the graph intersects the line $y = f(x)$ more than once, then f is not Single Valued, therefore f is not a function.

EXAMPLES:





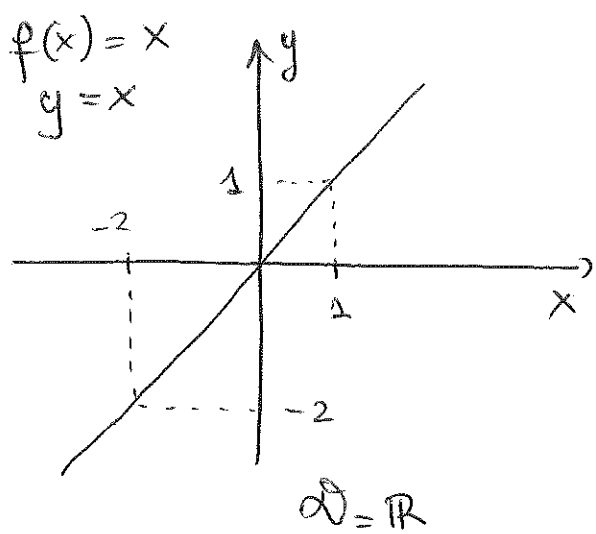
The most basic graphing technique is

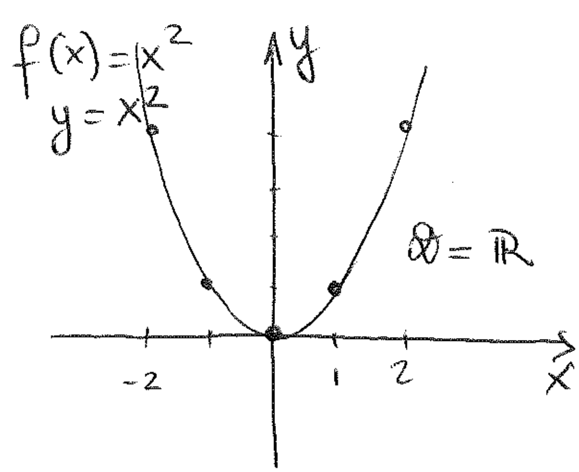
- to construct a table of values, with two columns: values of x and corresponding values of $f(x)$
- draw the corresponding points with coordinates $(x, f(x))$.

We have seen a few examples of this technique already. However, graphing point by point be a little bit tedious, and there are a lot of faster techniques. They are based on knowing the graphs of "standard" functions, and how mathematical and geometrical manipulations of these graphs relate to one another.

THE STANDARD FUNCTIONS

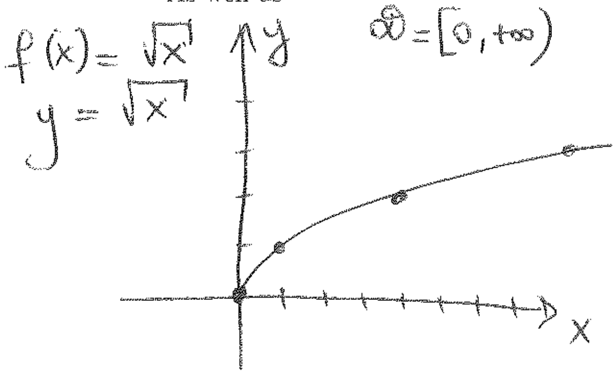
The following standard functions are ones everyone should know how to graph, accurately, "by heart", from now on. As the course proceeds, we will continue adding to this list of functions.



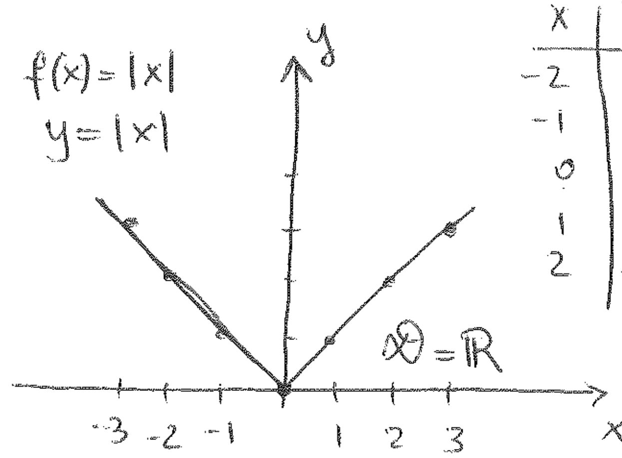


x	-2	-1	0	1	2
$y = x^2$	4	1	0	1	4

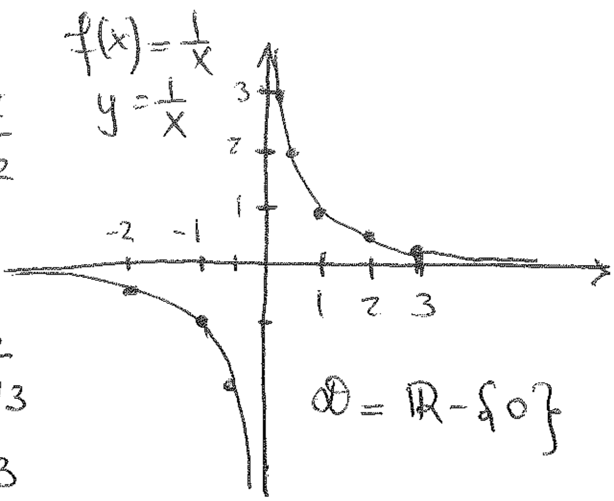
As well as



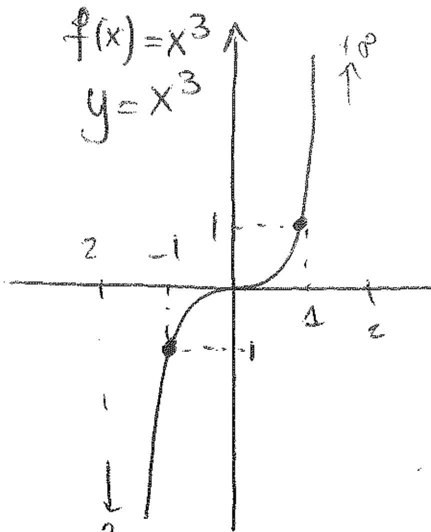
x	$\sqrt{x}$
0	0
1	1
4	2
9	3



x	$\lvert x \rvert$
-2	2
-1	1
0	0
1	1
2	2



x	$1/x$
-2	-1/2
-1	-1
1	1
2	1/2
-3	-1/3
3	1/3
$\frac{1}{2}$	2
$-\frac{1}{2}$	-2

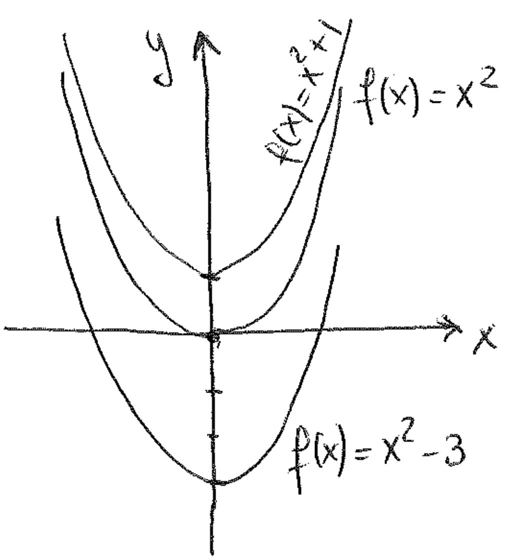


-2	-8
-1	-1
0	0
1	1
2	8

Simple operations on functions, and their geometric interpretation.

Once the graphs of basic functions are known, it is quite easy to find out what the graph of various other related functions may be. Let us go through a list of simple examples.

- *Adding or subtracting a number to a function.*
  Let's consider for instance the simple basic function $f(x) = x^2$, and then construct two other functions $g(x) = f(x) + 1 = x^2 + 1$ and $h(x) = f(x) - 3 = x^2 - 3$.

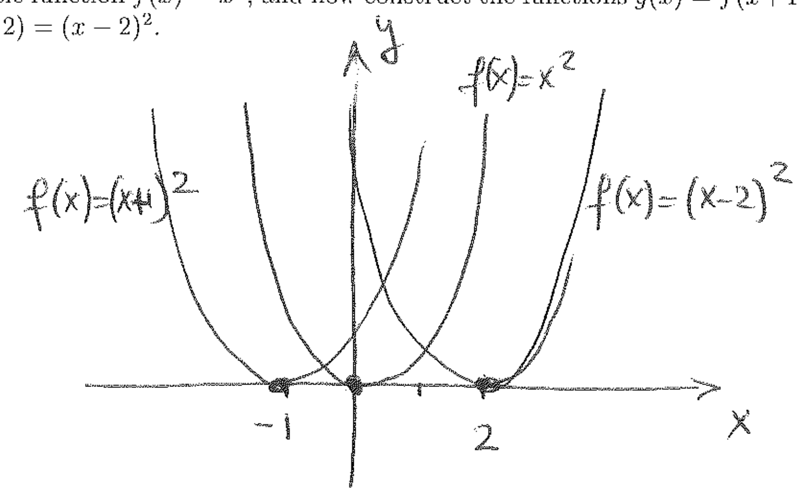


$f(x) = x^2 + 1$ has a graph that is shifted up by 1 compared to $f(x) = x^2$

$f(x) = x^2 - 3$ has a graph that is shifted down by 3 compared to $f(x) = x^2$

This can easily be generalized as : The graph of the function $g(x) = f(x) + a$ is shifed up by $|a|$ (if $a > 0$) or down (if $a < 0$) compared to the graph of $f(x)$

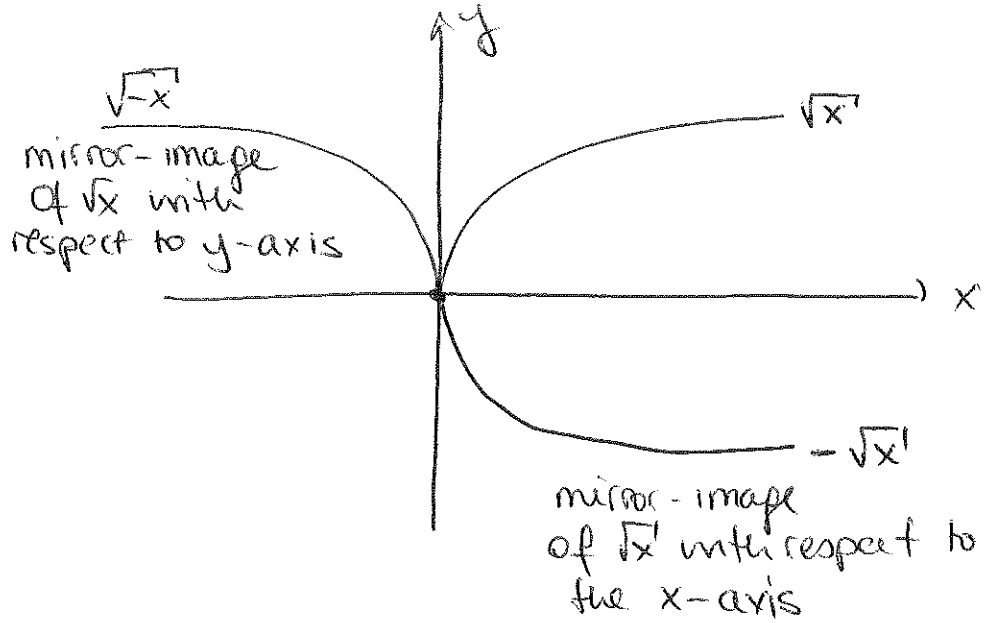
- *Applying a function to $x + a$ or $x - a$.*
  Let's consider again the basic function $f(x) = x^2$, and now construct the functions $g(x) = f(x+1) = (x+1)^2$ and $h(x) = f(x-2) = (x-2)^2$.



- The graph of $f(x) = (x-2)^2$ is shifted to the <u>right</u> by 2 compared to x^2
- The graph of $f(x) = (x+1)^2$ is shifted to the <u>left</u> by 1

Again, this can easily be generalized as: The graph of $g(x) = f(x+a)$ is obtained by shifting that of $f(x)$ by $|a|$ to the left (if $a > 0$) & ~~to~~ to the right if $a < 0$

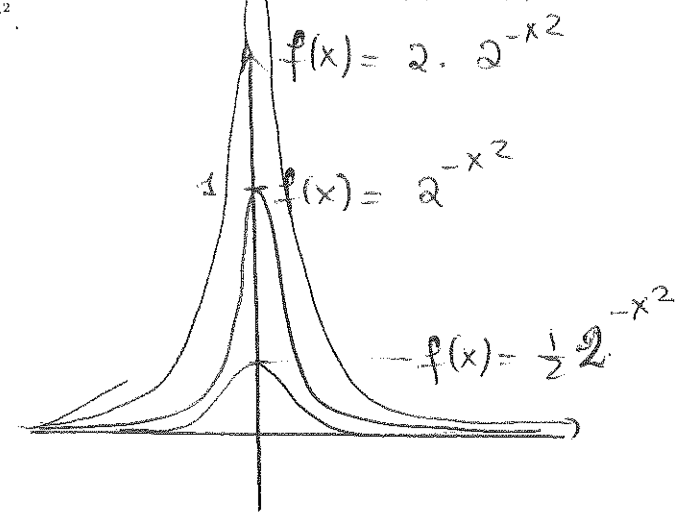
- *The functions $-f(x)$ and $f(-x)$.*
  This time, we will consider the basic function $f(x) = \sqrt{x}$, and construct $g(x) = -f(x) = -\sqrt{x}$, as well as $h(x) = f(-x) = \sqrt{-x}$.



This can easily be generalized as :

- The graph of the function $g(x) = -f(x)$ is the mirror image of the graph of $f(x)$ with respect to the x-axis
- The graph of the function $g(x) = f(-x)$ is the mirror image of the graph of $f(x)$ w.r.t the y-axis.

- *Scaling the dependent or independent variables.*
  To illustrate this case, let's consider another function that we will study in more detail later in this course: $f(x) = 2^{-x^2}$. We then use this function to construct both $g(x) = 2f(x) = 2 \times 2^{-x^2}$, as well as $h(x) = f(2x) = 2^{-(2x)^2} = 2^{-4x^2}$.



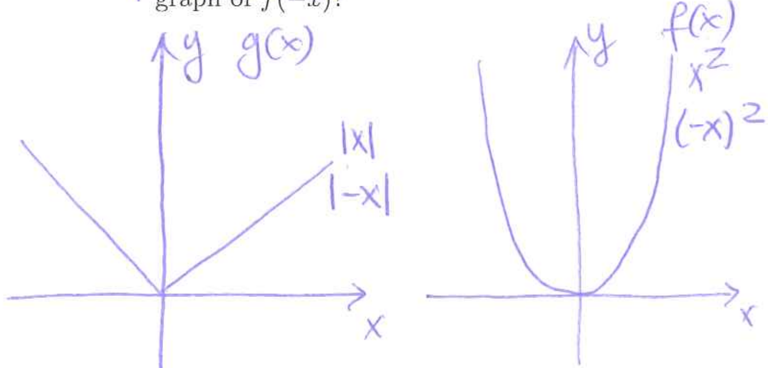
This can easily be generalized as :

The graph of the function $g(x) = a \cdot f(x)$ can be obtained by stretching or squeezing the graph of $f(x)$ vertically

EVEN AND ODD FUNCTIONS

It is worth noting that some graphs have interesting symmetries. For instance the graphs of $f(x) = x^2$ and $f(x) = |x|$ are mirror-symmetric across the y-axis, while the graphs of $f(x) = x^3$ and $f(x) = 1/x$ are point symmetric across the origin. Not all graphs have these symmetries, however. For instance $f(x) = \sqrt{x}$ does not have any special symmetry. Let us now see how the symmetry of the graph relates to the mathematical formula for the function.

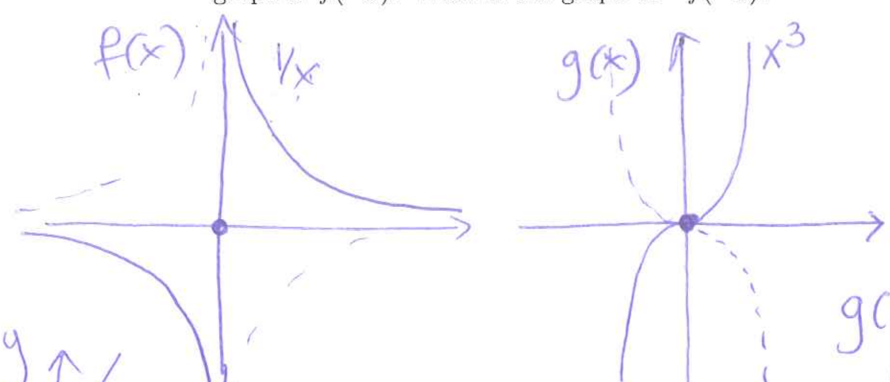
Even functions. Consider a function $f(x)$ whose graph is mirror-symmetric across the y-axis. What is the graph of $f(-x)$?



$f(-x) = (-x)^2 = x^2 = f(x)$
$g(-x) = |-x| = |x| = g(x)$

→ The functions $f(-x)$, $g(-x)$ have the same expression AND the same graph as $f(x)$ & $g(x)$

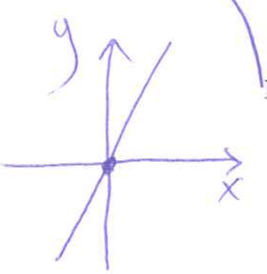
Odd functions. Consider a function $f(x)$ whose graph is point-symmetric across the origin. What is the graph of $f(-x)$? What is the graph of $-f(-x)$?



$f(-x) = \frac{1}{-x} = -\frac{1}{x} = -f(x)$

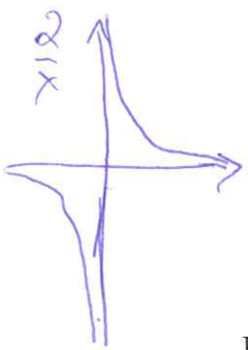
$g(-x) = (-x)^3 = -x^3 = -g(x)$

The functions $f(-x)$ and $g(-x)$ have the same graph as $-f(x)$ and $-g(x)$



EXAMPLES: Are these functions odd or even or neither?

- $f(x) = 3x$ → $f(-x) = 3(-x) = -3x = -f(x)$ → ODD
- $f(x) = -x + 1$ → $f(-x) = -(-x) + 1 = x + 1$ NEITHER!
- $f(x) = x^4$ → $f(-x) = (-x)^4 = (-1)^4 x^4 = x^4 = f(x)$ → EVEN



- $f(x) = -x^5$
- $f(x) = \frac{2}{x}$ → $f(-x) = \frac{2}{-x} = -\frac{2}{x} = -f(x)$ → ODD
- $f(x) = -\frac{4}{x^2}$

Knowing that a function is odd or even *before* graphing it is quite useful: if we do, then we only have to graph it precisely for $x > 0$ (or $x < 0$), and then we can find out what the rest of the function looks like simply by symmetry.