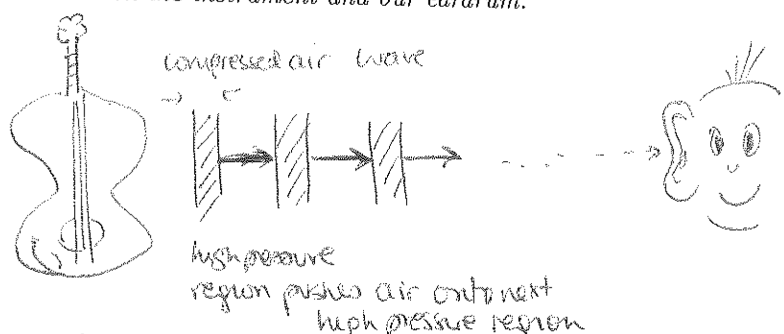
6.5 Trigonometric formulas

Textbook Section 6.4

6.5.1 Case Study: *The beating phenomenon in music*

Watch video on sound beating: https://www.youtube.com/watch?v=IYeV2Wq82fw

As we see in the video, two musical notes played at the same time but with slightly different tones interfere with one another and produce a phenomenon called beating. This study will help us understand why this happens. First, note that a sound is actually a pressure wave, i.e. a compressional oscillation of the air between the instrument and our eardrum.



Because of this the mathematical equation that describes one sound wave is a simple oscillatory function. Example:

pressure as function of time $\quad p(t) = a\sin(bt) + p_0$ ← basic atmospheric pressure

The frequency of the oscillation b is directly related to the pitch of the sound: low frequency sounds are low-pitched sounds, while high frequency sounds are high-pitched sounds.

Middle A note : 440 Hz $\qquad$ 1 Hz = 1 oscillation/second.

Middle C note : 261.6 Hz

Meanwhile, the amplitude of the oscillation a is related to how loud the sound it: high amplitude sound is loud, while low amplitude is quiet.

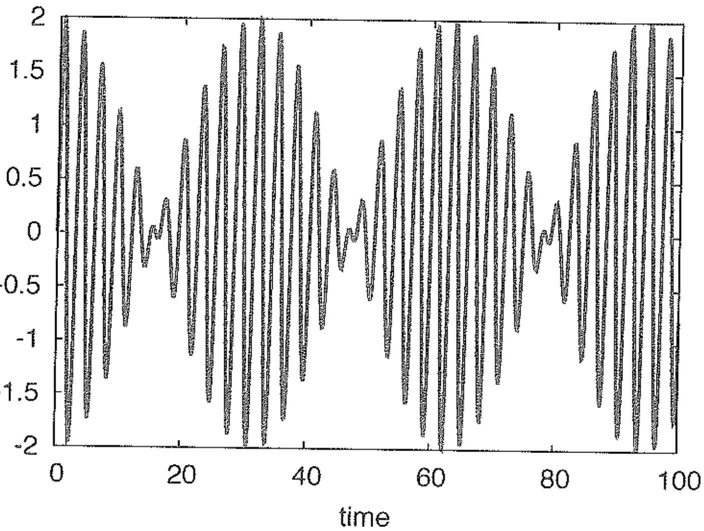
When an instrument plays together two notes of different pitch, but similar amplitude then the equation for the sum of the two waves is simply:

$$p(t) = p_1(t) + p_2(t) = a\sin(b_1 t) + a\sin(b_2 t) + p_0.$$

↑ frequency of first note $\qquad$ ↑ frequency of second note

Suppose now, as suggested in the video, that the two waves have exactly the same amplitude, and very similar frequencies. For instance, let's pick $b_1 = 2$ and $b_2 = 2.2$, and graph the resulting sum of two waves:
(Assume $a = 1$)

$$p(t) = \sin(2t) + \sin(2.2t)$$



The sum of the two waves is an oscillation with varying amplitude (ranging from 0 to 2)

→ this means the loudness of the sound goes up & down, as in the video.

Beating phenomenon.

To understand the phenomenon more generally, we will need to learn a few trigonometric formula first.

6.5.2 Trigonometric formulae

There are a few very important formulas in trigonometry. You will only need to know two of them by heart for this class, but you will need to know how to use the other ones if they are provided to you.

THE BASIC FORMULAS: (must be known by heart)

We've already seen the two basic formulas as part of the previous lectures; these must be known by heart.

- $\tan x = \frac{\sin x}{\cos x}$ Definition of tan x
- $\sin^2 x + \cos^2 x = 1$ Pythagorean formula

It is important to learn to use these formulas to simplify various expressions.

EXAMPLES OF USE:

- Simplify $\sin(x) - \sqrt{1-\cos^2 x}$ (assume x is between 0 and pi).

$1 - \cos^2 x = \sin^2 x$ so

$\sin x - \sqrt{1-\cos^2 x} = \sin x - \sqrt{\sin^2 x} = \sin x - |\sin x| = 0$

if $\sin x > 0$ (which it is)

- Simplify: $1 + \tan^2(x)$

$1+\tan^2(x) = 1 + \frac{\sin^2 x}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$

$(a+b)(a-b) = a^2 - b^2$

$\theta =$ theta

- Prove that $\frac{1-\sin\theta}{\cos\theta} = \frac{\cos\theta}{1+\sin\theta}$ → cross multiply

$$(1-\sin\theta)(1+\sin\theta) = \cos\theta \cdot \cos\theta = \cos^2\theta$$

true as well

$$1 - \sin^2\theta = \cos^2\theta$$

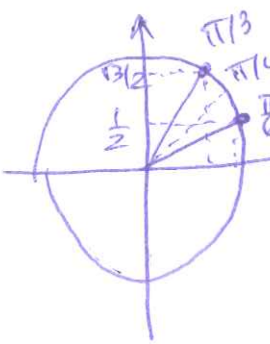
$$1 = \sin^2\theta + \cos^2\theta$$ ← true because it's Pythagorean formula

The list of trigonometric identities which can be proved using the basic formulas is endless. See Textbook Examples 1-8 for instance.

THE ADDITION FORMULAS (don't need to know by heart).

The addition formulas relate the sines and cosines of *sums* of angles to the *products* of sines and cosines of basic angles. The 4 formulas are

- $\cos(a+b) = \cos a \cos b - \sin a \cdot \sin b$
- → $\cos(a-b) = \cos a \cos b + \sin a \sin b$
- $\sin(a+b) = \sin a \cos b + \sin b \cos a$
- $\sin(a-b) = \sin a \cos b - \sin b \cos a$.



EXAMPLES: These examples show that these formulas indeed work:

- $\cos(\pi/3 - \pi/6) = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$ $\qquad \frac{\pi}{3} - \frac{\pi}{6} = \frac{2\pi}{6} - \frac{\pi}{6} = \frac{\pi}{6}$

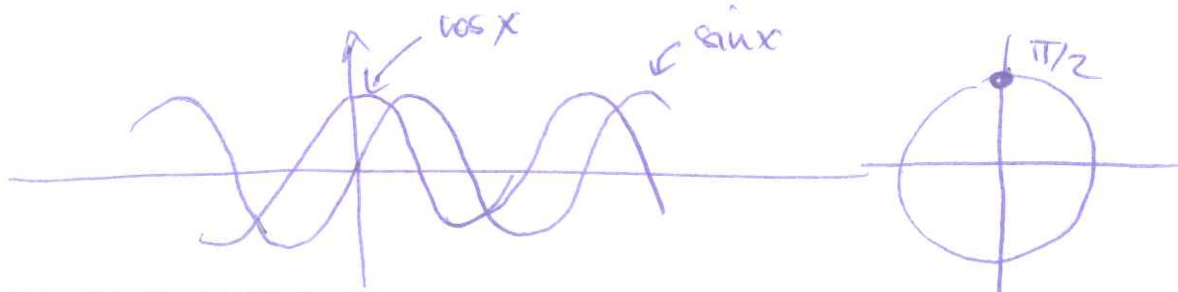
$$= \cos\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{6}\right) + \sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\frac{1}{2} = \frac{\sqrt{3}}{2}$$

- ~~$\cos(\pi/3 + \pi/3)$~~
- $\cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \cos\left(\frac{\pi}{12}\right)$
- ~~$\sin(\pi/4 + \pi/4)$~~

$$= \cos\frac{\pi}{3}\cos\frac{\pi}{4} + \sin\frac{\pi}{3}\sin\frac{\pi}{4}$$

$$= \frac{1}{2}\cdot\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2}$$

$$= \frac{\sqrt{2}}{2}\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\right) = \frac{\sqrt{2}}{4}(1+\sqrt{3})$$



Also, they confirm our hunch that $\sin(x)$ and $\cos(x)$ are the "same" function but displaced by $\pi/2$. Indeed, we saw from the graph that $\boxed{\sin(x) = \cos(x - \pi/2)}$. We can now prove this mathematically:

$$\cos\left(x - \frac{\pi}{2}\right) = \cos x \cos\frac{\pi}{2} + \sin x \sin\frac{\pi}{2} = \cancel{\cos x \cdot 0} + \sin x \cdot 1$$
$$= \sin x$$

NOTE: There is no equivalent *product* formulas: there is no simple identity for $\cos(ab)$ and $\sin(ab)$.

Altogether these identities can be used to prove another nearly-infinite number of identities.

skip

EXAMPLE 1: Prove that $\cos^2 a - \sin^2 b = \cos(a - b)\cos(a + b)$

EXAMPLE 2: Prove that $\sin^2 a - \sin^2 b = \sin(a - b)\sin(a + b)$

THE DOUBLE-ANGLE FORMULAS (know by heart)

As a consequence of the addition formulas, we have 3 more formulas which are called the *double-angle* formulas because they express the sine and cosine of the angle $2a$ in terms of the sine and cosine of the angle a. These formulas are

- $\sin(2x) = 2\sin x \cdot \cos x$
- $\cos(2x) = \cos^2 x - \sin^2 x$

To show them, note that

$$\cos(2x) = \cos(x+x) = \cos x \cos x - \sin x \sin x = \cos^2 x - \sin^2 x$$

$$\sin(2x) = \sin(x+x) = \sin x \cos x + \sin x \cos x = 2 \sin x \cos x$$

Again, these double-angle formula can be used to simplify trigonometric expressions.

EXAMPLE 1: Simplify $\frac{\sin(2x)}{\cos(x)}$

$$\frac{\sin 2x}{\cos x} = \frac{2 \sin x \cancel{\cos x}}{\cancel{\cos x}} = 2 \sin x$$

EXAMPLE 2: Show that $\cos(2x) + 1 = 2\cos^2(x)$

$$\cos(2x) + 1 = \cos^2 x - \sin^2 x + 1 = \cos^2 x + \underbrace{1 - \sin^2 x}_{\cos^2 x}$$
$$= 2\cos^2 x$$

6.5.3 Case study: *The beating phenomenon (part 2)*

Let's now consider generally two sound waves, one with frequency $b + s$ (where s is a small number), and one with a frequency $b - s$.

$$p(t) = a \sin((b-s)t) + a \sin((b+s)t)$$

For instance, our two waves earlier can be cast in this form, by letting

$$\begin{aligned} 2 &= b_1 = b - s \\ 2.2 &= b_2 = b + s \end{aligned} \quad \rightarrow \text{works if } \begin{cases} b = 2.1 \\ s = 0.1 \end{cases}$$

Let's now add these two waves, and use the sum formula. We have

$$\begin{aligned} p(t) &= a \sin((b-s)t) + a \sin((b+s)t) \\ &= a[\sin(bt - st) + \sin(bt + st)] \\ &= a[\sin bt \cos st - \cancel{\cos bt \sin st} + \sin bt \cos st + \cancel{\cos bt \sin st}] \\ &= 2a \sin bt \cos st \end{aligned}$$

This shows that the sum of these two waves is also equal to the product of two waves, one whose frequency is b (a high frequency that is half-way between the two original ones), and one whose frequency is s (a low frequency, which is half the difference between the two original ones). When a low frequency signal *multiplies a high-frequency one, the resulting function looks like the high-frequency oscillation, but instead of having a constant amplitude, the amplitude varies in time according to the low-frequency signal. This is exactly what we are seeing here. The low frequency signal modulates the amplitude of the high frequency one, and causes the beating phenomenon. This low frequency is therefore called the beat frequency.*

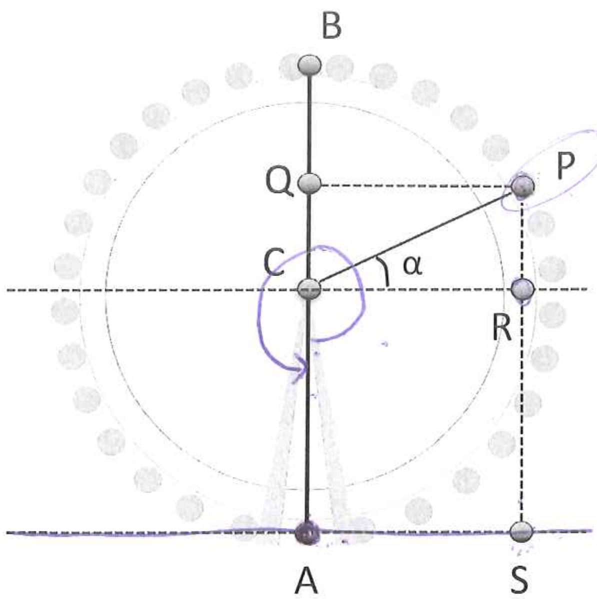
6.6 Solving trigonometric equations

Textbook Section 6.3

6.6.1 Case Study: *The London Eye*

The London Eye is a huge Ferris Wheel whose radius is 200ft. It is shown in the diagram below. Going once all the way around at a steady pace takes about 30 minutes, from start to finish. In order to take a good picture of the London sights, you need to be at least 300 ft above the ground. How long will you have to take pictures while being above that height?

ground



CA = 200ft

CB = ...200...

CP = ...200...

RS = ...200...

AB = ...400...

The next 2 answers depend on the angle α

PR = ...200 sin α...

PS = ...PR + RS... $= 200\sin\alpha + 200$

$h(\alpha) = 200(1+\sin\alpha)$

$\rightarrow \frac{PR}{PC} = \sin\alpha \Rightarrow PR = 200\sin\alpha$

In order to answer this question, we need to do a little bit of mathematical modeling. Suppose you board the London Eye at the point A, and begin to circle around it in a counterclockwise manner. Your position in the Eye at a later time is the point P, and that point circles the Eye as time goes by. The points A, B, C are fixed, but the points P, Q (the projection of P onto the axis of the wheel), R and S (the projection of P onto the ground) depend on where you are at that time, which depends on the angle α shown.

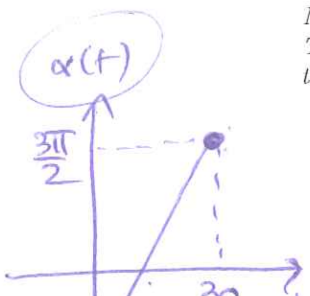
We see from this diagram that your height above the ground is the distance:

As time goes by and the point P circles the Eye, this distance first increases, then decreases. There will be a short interval of time where the height above the ground exceeds 250ft, and we would like to know what that interval is.

In order to answer that question, we first need to do a little geometry. Let's write the distance PS as a function of the angle α.

(see above)

Next we must use the information from the text that the wheel goes around at a steady pace in 30 minutes. This means that the angle α increases with time linearly, from $-\pi/2$ at time $t = 0$, to $3\pi/2$ at time $t = 30min$. We can construct a linear function $\alpha(t)$ that fits this information:



at time $t=0$ $\alpha = -\frac{\pi}{2}$ at $t=30$, $\alpha = \frac{3\pi}{2}$

How to create a linear function $\alpha(t)$ such that $\alpha(0) = -\frac{\pi}{2}$ and $\alpha(30) = +\frac{3\pi}{2}$?

Slope $m = \dfrac{\frac{3\pi}{2} - (-\frac{\pi}{2})}{30} = \dfrac{2\pi}{30}$ $\boxed{\alpha(t) = \frac{2\pi}{30}t - \frac{\pi}{2}.}$

Putting this together with the height formula, we find that a person's height above the ground when travelling in the wheel is the following function of time:

$$h(t) = h(\alpha(t)) = 200\left(1 + \sin(\alpha(t))\right)$$
$$= 200\left(1 + \sin\left(\frac{2\pi t}{30} - \frac{\pi}{2}\right)\right)$$
$$h(t) = 200 + 200\sin\left(\frac{2\pi t}{30} - \frac{\pi}{2}\right)$$

Finally, we want to know how long a person will be above 300 ft. In order to do that, we have to find the two times at which the person is exactly 300 ft up, and take the difference between the two. This first requires solving the following mathematical equation:

$h(t) = 300$? at what times is this true?

$$\cancel{200} + 200\sin\left(\frac{2\pi t}{30} - \frac{\pi}{2}\right) = 300 - 200 = 100$$

$$\cancel{200}\sin\left(\frac{2\pi t}{30} - \frac{\pi}{2}\right) = \frac{100}{200} = \frac{1}{2}$$

$\rightarrow$ solve $\left[\sin\left(\frac{2\pi t}{30} - \frac{\pi}{2}\right)\right] = \frac{1}{2}$ for t.

This is a trigonometric equation, i.e. an equation that involves trigonometric functions. We will now learn how to solve them.

6.6.2 Solving basic trigonometric equations

Trigonometric equations are equations that involve trigonometric functions, and that need to be solved for the unknown variable. There are many different kinds of trigonometric equations, though most of them ultimately require you to either solve $\cos(x) = a$ (for a given a), or solve $\sin(x) = a$ (for a given a), or solve $\tan(x) = a$ (for a given a).

The tricky thing about trigonometric equations is that sometimes they do not have solutions, but when they do have solutions, they often have infinitely many of them – and all of them need to be found. Let's work by examples to see what may happen.

EXAMPLE OF EQUATIONS THAT DO NOT HAVE SOLUTIONS.

- $\cos(4x + 2) = 5$.

  Does not have a solution because cos(any number) lies between -1 & 1

- $\cos^2 x + \sin^2 x = 2$.

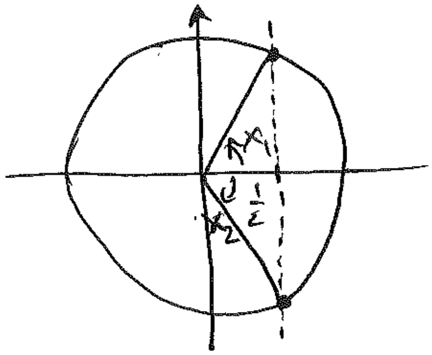
  Does not have a solution because $\cos^2 x + \sin^2 x = 1$ and 1 cannot be equal to 2!

- $\cos^2 y + 2\sin^2 y = 3$.

  This can be rewritten $\cos^2 y + \sin^2 y + \sin^2 y = 3$
  $\Rightarrow 1 + \sin^2 y = 3 \Rightarrow \sin^2 y = 2 \rightarrow$ not possible
  (because $\sin^2 y$ lies between 0 and 1)

EXAMPLES OF BASIC TRIGONOMETRIC EQUATIONS THAT HAVE INFINITELY MANY SOLUTIONS

- Solve the equation $\cos(x) = \frac{1}{2}$



Use unit circle, & find point on the circle whose <u>x-coordinate</u> is $\frac{1}{2}$

$\rightarrow$ This defines the angles

$$x_1 = \frac{\pi}{3}$$
$$x_2 = -\frac{\pi}{3}$$

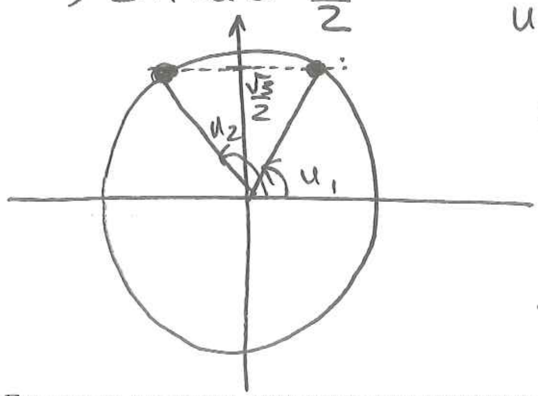
as well as all further angles " $+ 2\pi n$ "

$\rightarrow$ all solutions are

$$x = \left\{ \frac{\pi}{3} + 2\pi n, \ -\frac{\pi}{3} + 2\pi n \right\}$$

- Solve the equation $\sin(3x) = \frac{\sqrt{3}}{2}$ $\rightarrow$ use intermediate variable $u = 3x$

$\rightarrow \sin u = \frac{\sqrt{3}}{2}$



$u_1 = \frac{\pi}{3} + 2n\pi \qquad u_2 = \frac{2\pi}{3} + 2\pi n$

since $u = 3x \rightarrow x = \frac{u}{3}$

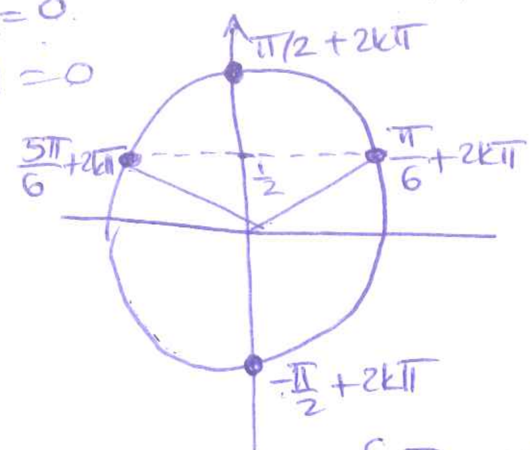
$$\begin{cases} x_1 = \frac{u_1}{3} = \frac{1}{3}\left(\frac{\pi}{3} + 2n\pi\right) \\ x_2 = \frac{u_2}{3} = \frac{1}{3}\left(\frac{2\pi}{3} + 2n\pi\right) \end{cases}$$

EXAMPLES OF MORE ADVANCED TRIGONOMETRIC EQUATIONS THAT REQUIRE USING SOME TRIGONOMETRIC FORMULAS FIRST

$x^3 - x = 0$

$x^2 - 1 = 0$

- Solve the equation $\sin(2x) = \cos(x)$ $\rightarrow$ Solve $2\sin x \cos x = \cos x$



$\rightarrow 2\sin x \cos x - \cos x = 0$

$\rightarrow \cos x\,(2\sin x - 1) = 0$

$\rightarrow$ either $\begin{cases} \cos x = 0 \\ \text{or } 2\sin x - 1 = 0 \end{cases}$

$\left(\sin x = \frac{1}{2}\right)$

$\rightarrow \left\{ \frac{\pi}{2} + 2k\pi, -\frac{\pi}{2} + 2k\pi, \frac{\pi}{6} + 2k\pi, \frac{5\pi}{6} + 2k\pi \right\}$.

- Solve the equation $2\cos^2(a) + 1 = 2\sin^2(a)$

$2\cos^2 a + 1 = 2\sin^2 a \Rightarrow 2(\cos^2 a - \sin^2 a) = -1$

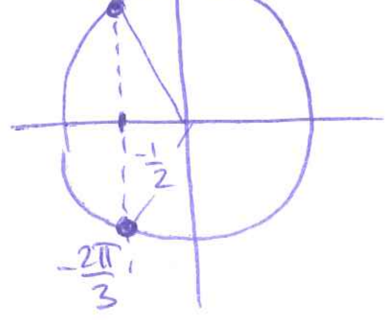
$\Rightarrow 2\cos(2a) = -1$

let $u = 2a$

$\Rightarrow \cos(2a) = -\frac{1}{2}$

$a = \frac{u}{2}$

$\Rightarrow \cos u = -\frac{1}{2}$

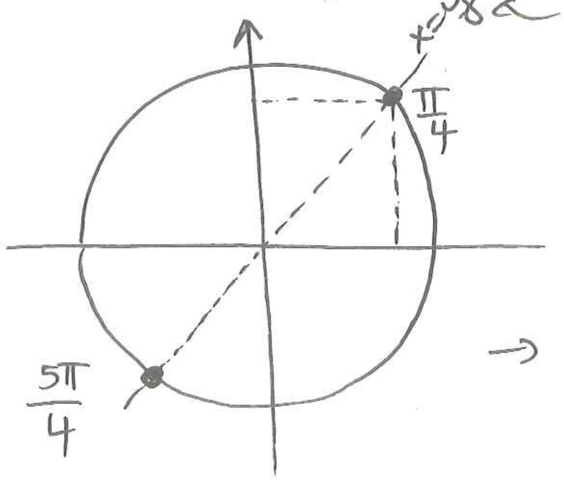


$u_1 = \frac{2\pi}{3} + 2\pi k \qquad u_2 = -\frac{2\pi}{3} + 2\pi k$.

$a_1 = \frac{1}{2}\left(\frac{2\pi}{3} + 2\pi k\right) \qquad a_2 = \frac{1}{2}\left(-\frac{2\pi}{3} + 2\pi k\right)$

EXAMPLES OF TRIGONOMETRIC FUNCTIONS THAT CAN BE SOLVED GRAPHICALLY

- Solve the equation $\boxed{\sin(a) = \cos(a)}$



on this x=y line all the points have equal x-coordinate & y-coordinate.

→ on that line, points have equal sine & cosine

$\rightarrow \left\{ \frac{\pi}{4} + 2n\pi \text{ and } \frac{5\pi}{4} + 2n\pi \right\}$

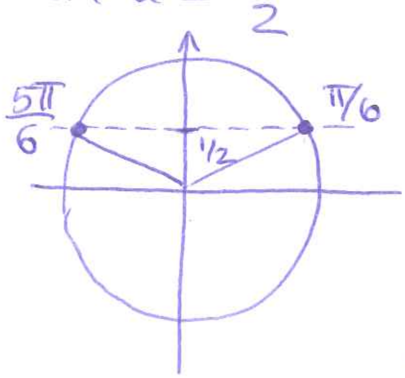
Let's now go back to the case study and solve the problem.

6.6.3 Case Study: *The London Eye (part 2)*

We can now solve the previously derived equations to find at what time(s) the wheel is exactly 300ft up:

$\sin\left(\frac{2\pi t}{30} - \frac{\pi}{2}\right) = \frac{1}{2} \quad \rightarrow$ use $u = \frac{2\pi t}{30} - \frac{\pi}{2}$

$\sin u = \frac{1}{2} \quad \rightarrow$ solutions are $u = \frac{\pi}{6}$ and $\frac{5\pi}{6}$



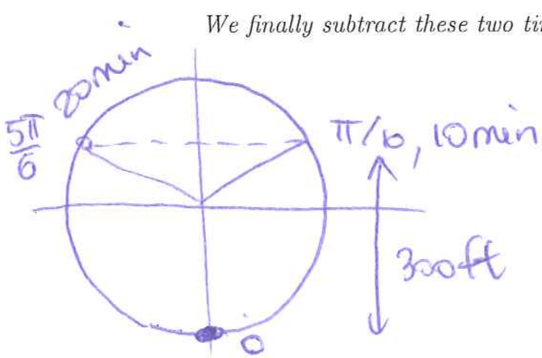
→ Solve for t

$$\frac{2\pi t}{30} = u + \frac{\pi}{2}$$

$$t = \frac{30}{2\pi}\left(u + \frac{\pi}{2}\right)$$

$$\rightarrow t_1 = \frac{30}{2\pi}\left(\frac{\pi}{6} + \frac{\pi}{2}\right) = \frac{30}{2\cancel{\pi}} \cdot \frac{4\cancel{\pi}}{6} = 10 \text{ min}$$

We finally subtract these two times to find how long a person will be above 300ft:



$$t_2 = \frac{30}{2\pi}\left(\frac{5\pi}{6} + \frac{\pi}{2}\right) = \frac{30}{2\cancel{\pi}} \frac{8\cancel{\pi}}{6} = 20 \text{ min}$$

→ We will be above ~~the ground~~ 300ft above exactly 10 mins.