Chapter 6

Trigonometric functions and periodic functions

In this final chapter of our course, we will learn about the three basic trigonometric functions, sine, cosine and tangent, as well as their use in geometry and in modeling oscillatory periodic functions.

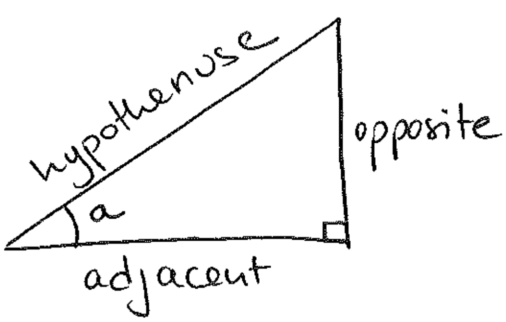
6.1 The basic trigonometric functions

6.1.1 Case study: *How to measure the heights of trees?*

The research group of Prof. Gilbert at UC Santa Cruz studies (among other things) the ecology of trees. One of the most crucial part of this research is the acquisition of data on the growth rates of various species of trees, which involves measuring the heights of trees at regular intervals. Now, while it is easy to measure the height of young saplings, how does one measure the height of ancient redwoods? Climbing them and using a measuring tape is certainly not an option. As it turns out, this problem is actually very easy provided one knows a little bit about the basic trigonometric functions. Let's now study them, and revisit the problem shortly.

6.1.2 Mathematical corner: Sine, cosine and tangent in right-angle triangles

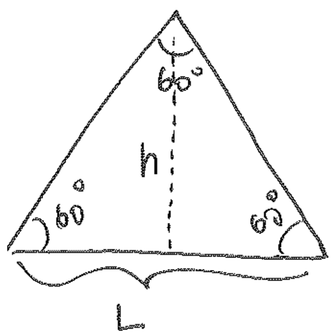
Sine, cosine and tangent functions are usually defined through their association with right-angle triangles:



$$\begin{cases} \sin a = \dfrac{\text{length of opposite}}{\text{length of hypothenuse}} \\[2ex] \cos a = \dfrac{\text{length of adjacent}}{\text{length of hypothenuse}} \\[2ex] \tan a = \dfrac{\text{length of opposite}}{\text{length of adjacent}} = \dfrac{\sin a}{\cos a} \end{cases}$$

The sine, cosine and tangent functions of various angles are easily computed using a calculator (also, see later for more). This knowledge can then be used to infer the size of one side knowing another side and an angle, as in the following examples.

EXAMPLE 1: What is the relationship between the height and the base of an equilateral triangle? Use this to deduce what the cosine of 60° is.

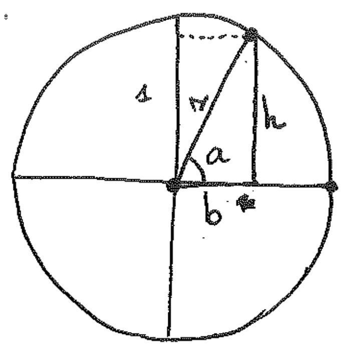


L is known, so what is h?

$$\tan 60^\circ = \frac{\text{opposite}}{\text{adjacent}} = \frac{h}{\frac{L}{2}} = 1.732...$$

$$\rightarrow h = \frac{L}{2}(1.732)... = 0.866\,L$$

EXAMPLE 2: Consider the following right-angle triangle inscribed in the unit circle. Using the Pythagorean formula, show that, for any angle a, $\cos^2 a + \sin^2 a = 1$.



Pythagoras: $b^2 + h^2 = 1^2$

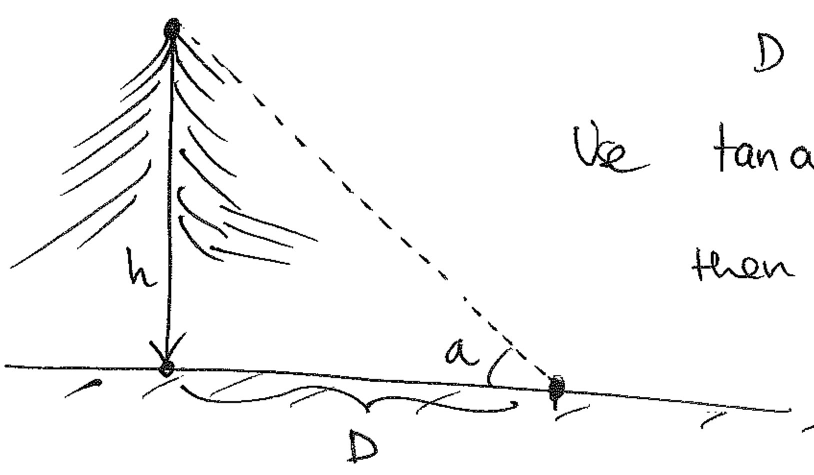
$$\sin a = \frac{h}{1} = h$$

$$\cos a = \frac{b}{1} = b$$

$$\boxed{(\cos a)^2 + (\sin a)^2 = 1}$$ Pythagorean Identity.

6.1.3 Case study: *How to measure the heights of trees? (part 2)*

We can now use what we have learned to measure the height of trees! Indeed, consider a tree, and walk a reasonable distance away from it so you can see the top. In as much as possible, try to do this horizontally (i.e. do not walk uphill or downhill). Measure the distance between where you are standing, and the base of the tree. Then, using a compass, measure the angle between the horizon and the top of the tree. We can then measure the tree height using:



D known, a known

Use $\tan a = \frac{\text{opposite}}{\text{adjacent}} = \frac{h}{D}$

then solve for $\boxed{h = D \cdot \tan a}$

For instance, what is the height of a redwood tree, if the angle measured 20 meters away from its base is 76°?

$a=76$ $\quad h = 20\tan(76^\circ) = 80.21$ meters. $\quad D=20$

6.2 Degrees and radians

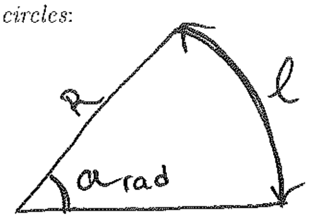
Textbook Section 5.1

6.2.1 Mathematical corner: Arc lengths, degrees and radians

There are two major ways of measuring angles in geometry: in *degrees* and in *radians*.

The degree measure was introduced historically in astronomy to measure the displacements of stars, and is based on the fact that there are approximately 360 days in a year (well, there are in fact 365.25 days in a year, but 360 conveniently divides nicely by 2, 3, 4, 6, 10, 12, .., while 365.25 doesn't).

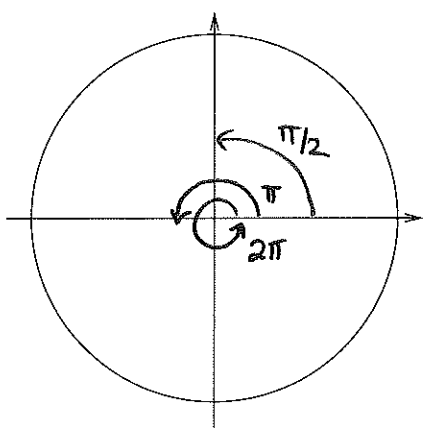
The radian measure is the one more commonly used in mathematics. It is defined from the length of *arcs of circles*:



l = distance along the circle covered by angle a

The radian measure of an angle is $\boxed{a_{rad} = \frac{l}{R}}$

Based on this we have the following correspondance between degree and radians:



- Going all the way around:
  $l = 2\pi R \Rightarrow a_{rad} = \frac{2\pi R}{R} = 2\pi$
  $\rightarrow$ 2π radians is 360°
- Going half way around
  $l = \pi R \rightarrow a_{rad} = \frac{\pi R}{R} = \pi$
  $\rightarrow$ π radians is 180°
- $90^\circ \leftrightarrow \frac{\pi}{2}$ $\quad 60^\circ \leftrightarrow \frac{\pi}{3}$
  $45^\circ \leftrightarrow \frac{\pi}{4}$ $\quad 30^\circ \leftrightarrow \frac{\pi}{6}$

$$a_{rad} = \frac{\pi}{180} a_{degr.}$$

$$a_{deg} = \frac{180}{\pi} a_{rad}$$

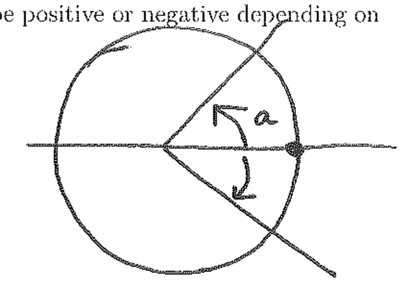
To summarize, to go between radians and degrees and vice-versa,

$$a_{rad} = \frac{\pi}{180} a_{deg}$$

$$a_{deg} = \frac{180}{\pi} a_{rad}.$$

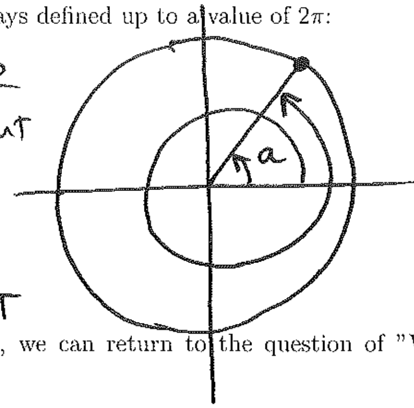
Note that while most calculators return the functions sine, cosine and tangent of an angle a, the user needs to input whether the angle is in degrees and radians.

Finally, note that for mathematical convenience angle are defined to be positive or negative depending on their direction:



- going counter clockwise the angle is POSITIVE
- going clockwise, the angle is negative.

Also note that since the circle wraps around, an angle is always defined up to a value of 2π:



The same point on the circle corresponds to many different angles:

$$a = a + 360^\circ = a - 360^\circ$$

$$a_{rad} = a_{rad} + 2\pi = a_{rad} - 2\pi$$

Now that we have introduced the concept of signed angles, we can return to the question of "What do the trigonometric functions look like?"

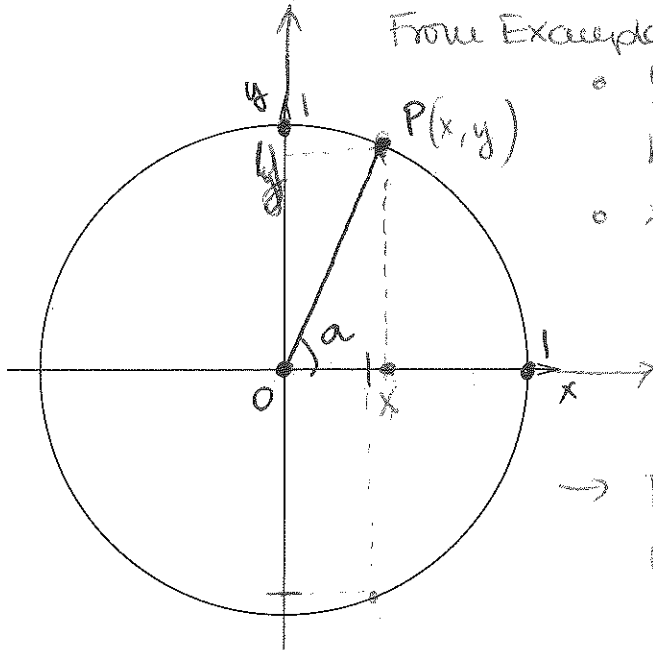
6.3 The unit circle, and the graphs of sine, cosine and tangent

Textbook Section 5.2

6.3.1 Construction of the unit circle

The unit circle is a wonderfully convenient way of *visualizing* the sine and cosine functions.

DEFINITION:
- The unit circle is the circle of radius 1 centered on the origin.
- Any point on the circle defines an angle (in radians or in degrees) up to an angle 2π or 360°



From Example 2 earlier we know that

- y is the height of triangle but is also sin a
- x is the base of the triangle but is also cos a

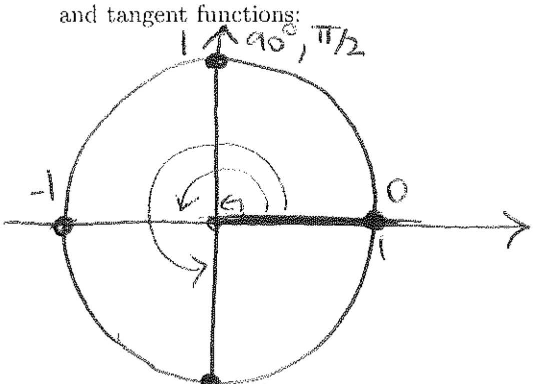
$$\begin{cases} x = \cos a \\ y = \sin a \end{cases}$$

→ The coordinates of the point P are $(\cos a, \sin a)$

→ The unit circle can be used to "visualise" the functions $\cos a$ and $\sin a$ (for any value of a).

6.3.2 Sine and Cosine of important angles

Based on the graph of the unit circle, we can already deduce some particular values of the sine, cosine and tangent functions:



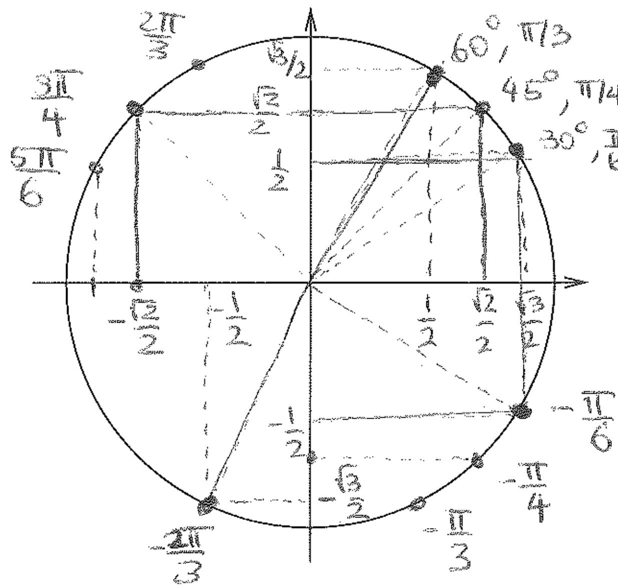
		x-coord	y-coord
$90^\circ, \frac{\pi}{2}$	:	$\cos\frac{\pi}{2} = 0$	$\sin\frac{\pi}{2} = 1$
$0^\circ, 0$	:	$\cos 0 = 1$	$\sin 0 = 0$
$180^\circ, \pi$	:	$\cos\pi = -1$	$\sin\pi = 0$
$270^\circ, \frac{3\pi}{2}$	:	$\cos\frac{3\pi}{2} = 0$	$\sin\frac{3\pi}{2} = -1$

In addition to 0, $\pi/2$, π, $3\pi/2$ and 2π, there are 3 important angles for which you need to know the sine and cosine of:

- $\frac{\pi}{6} \leftrightarrow 30^\circ$ $\quad \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$ $\quad \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$
- $\frac{\pi}{4} \leftrightarrow 45^\circ$ $\quad \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$ $\quad \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$
- $\frac{\pi}{3} \leftrightarrow 60^\circ$ $\quad \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$ $\quad \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

only 3 numbers to remember!

Based on the unit circle, we can now find the sine and cosine of many other angles:



By symmetry we see that

$$\sin\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$\cos\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$$\sin\left(-\frac{2\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

$$\cos\left(-\frac{2\pi}{3}\right) = -\frac{1}{2}$$

$$\cos\left(-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

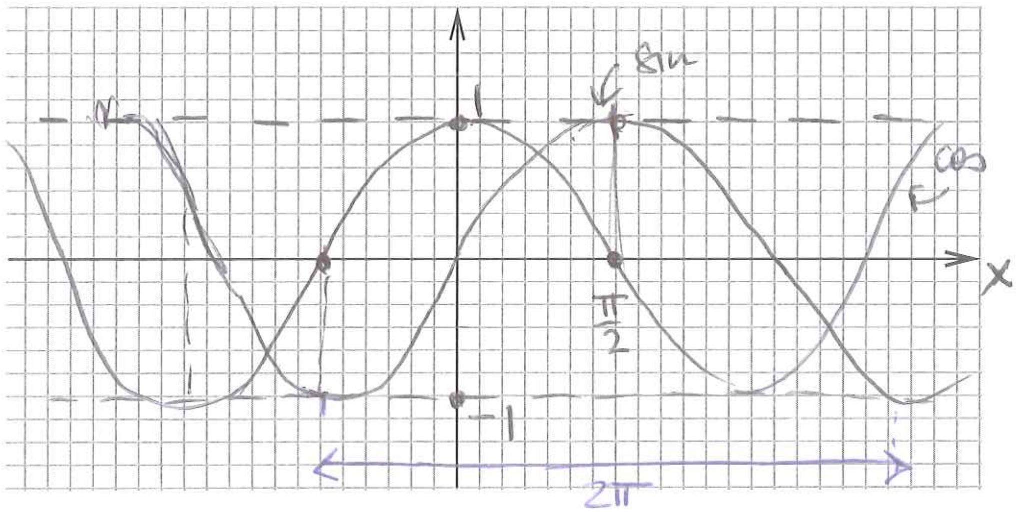
$$\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$$

a	$-\pi$	$-\frac{2\pi}{3}$	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
$\cos a$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	-1
$\sin a$	0	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	0
$\frac{\sin a}{\cos a} = \tan a$	0	$\sqrt{3}$	? ∞ ?	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	? ∞ ?	$-\sqrt{3}$	0
			asymptote								asymptote		

$$\frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = -\frac{\sqrt{3}}{\cancel{2}}\cdot\frac{\cancel{2}}{1} = -\sqrt{3}$$

$$\frac{-\frac{1}{2}}{\frac{\sqrt{3}}{2}} = -\frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{\sqrt{3}} = -\frac{1}{\sqrt{3}}$$

Finally, we can use this information to plot the sine and cosine functions:



6.3.3 What can we deduce from the graphs of $\sin(x)$ and $\cos(x)$?

Based on the graphs of $\sin(x)$ and $\cos(x)$, we see that

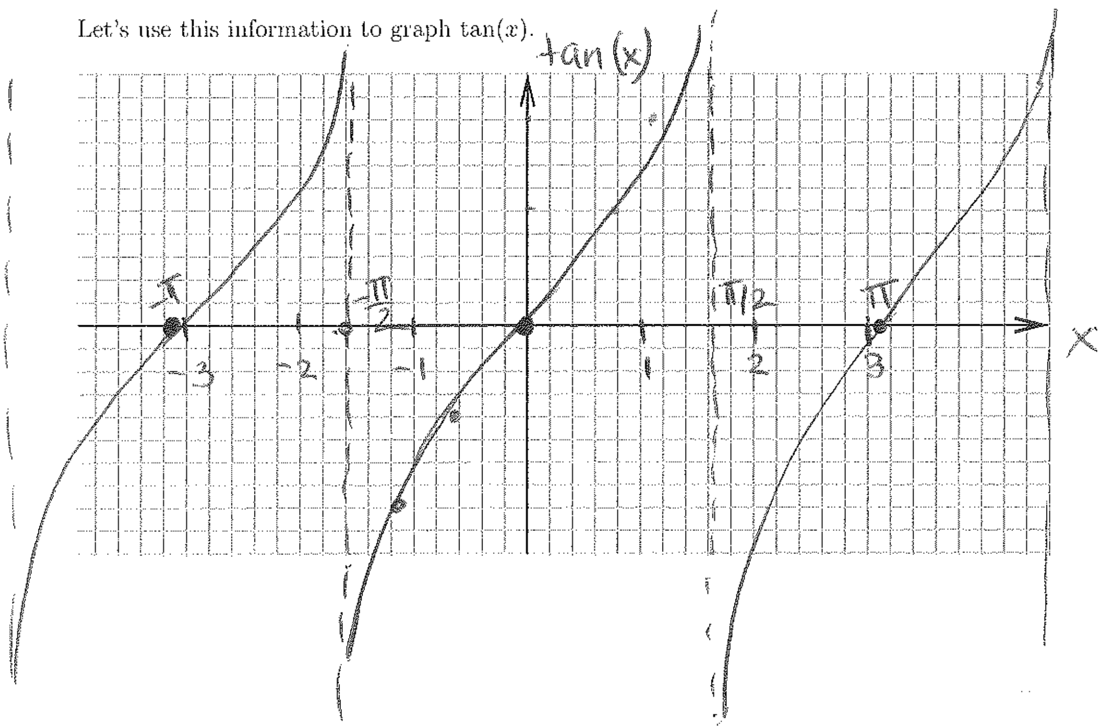
- sin(x) is odd and cos(x) is even.
- $-1 \leq \cos(x) \leq 1$ and $-1 \leq \sin(x) \leq 1$ for all x
- sin(x) is shifted by $\frac{\pi}{2}$ to the right compared with cos(x) → $\sin(x) = \cos\left(x - \frac{\pi}{2}\right)$
-   (phase shift property).
- $\sin(-x) = -\sin(x)$ $\cos(-x) = \cos(x)$

6.3.4 The graph of the tangent function

Textbook Section 5.5

We now look at the graph of the tangent function. By contrast with $\sin(x)$ and $\cos(x)$, $\tan(x)$ is not defined everywhere:

Let's use this information to graph tan(x).



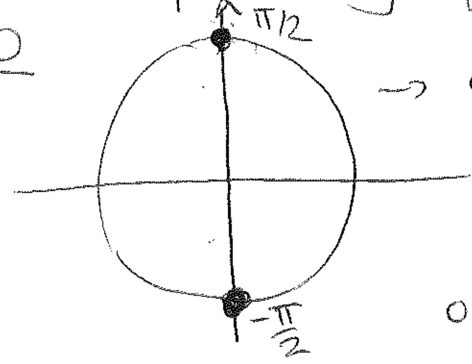
NOTES:

- The tangent function is odd → $\tan(-x) = -\tan(x)$
- It has asymptotes at $x = \frac{\pi}{2}, \frac{-\pi}{2}, -\frac{3\pi}{2}, \frac{3\pi}{2}$, etc ...
-

This comes from the fact that $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$

$$\tan(-x) = \frac{\overbrace{\sin(-x)}^{\text{odd}}}{\underbrace{\cos(-x)}_{\text{even}}} = \frac{-\sin(x)}{\cos(x)} = -\tan(x)$$

→ also we expect asymptotes whenever $\underline{\cos x = 0}$



→ asymptotes for any $x = \pm\frac{\pi}{2}$ + multiple of 2π

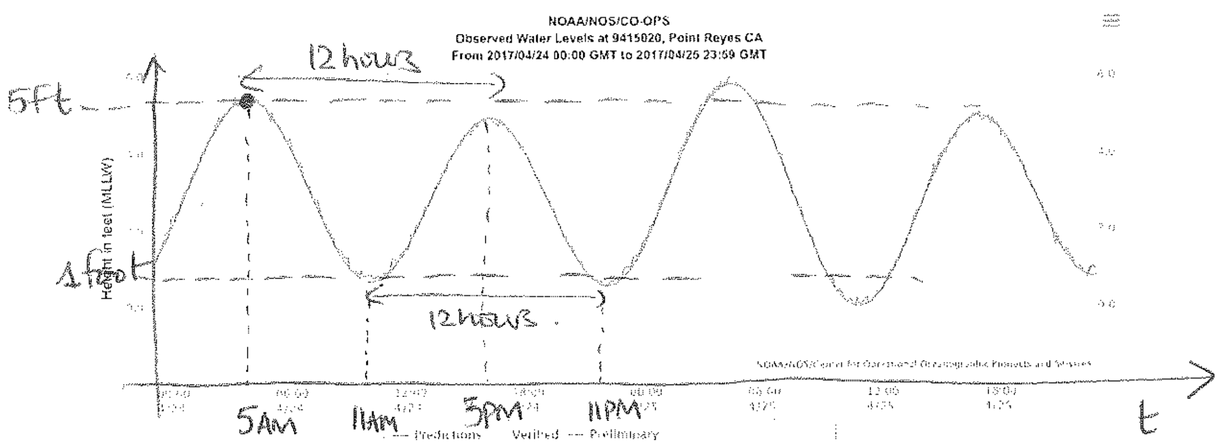
$$\text{or}\begin{cases} x = \frac{\pi}{2} + 2\pi(n) \\ x = -\frac{\pi}{2} + 2\pi n \end{cases}$$

6.4 Periodic functions

Textbook Section 5.6

6.4.1 Case Study: *Tides*

The National Oceanic and Atmospheric Administration studies the tidally-induced variation of the water level with time in various coastal areas around the US, including Point Reyes in CA. They provide both forecasting services (i.e. predictions of the future water level) and monitoring services (i.e. measuring the actual water level). The following figure shows the result of one of their predictions and monitoring efforts, for the 48-hour period starting at midnight on April 24th, 2017.

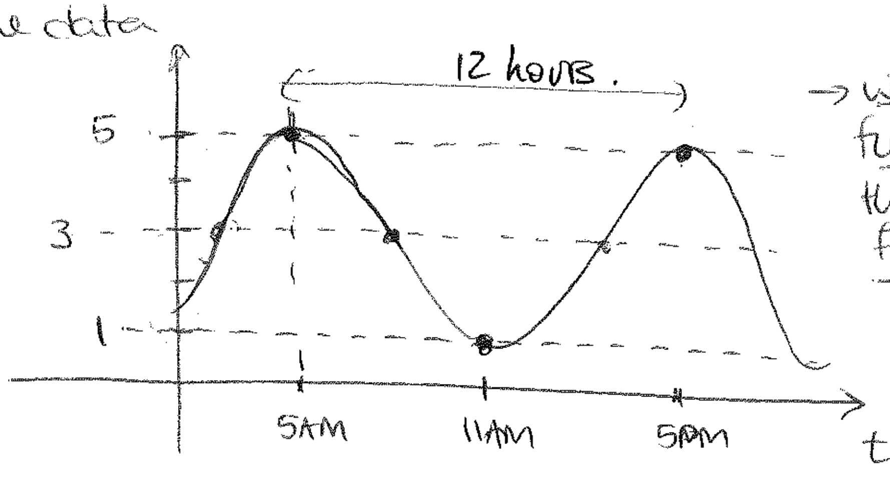


We see that

- The water level oscillates up & down every 12 hours
- low tides ~ 1ft & high tides ~ 5ft.

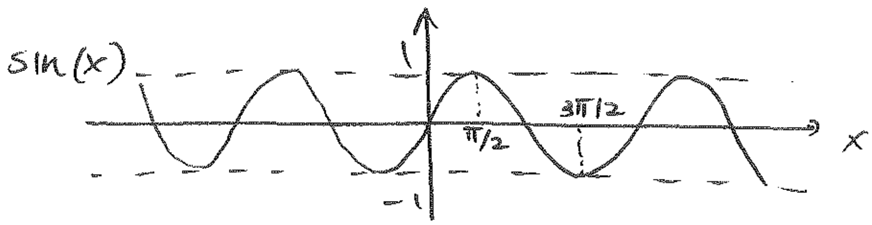
While it is actually quite difficult to model this phenomenon as accurately as NOAA did, we can try to model it approximately using sine or cosine functions. Let's relabel the hours starting at time $t = 0$, and increasing monotonically up to 48. Once that is done, how can we create a function that approximately models the data?

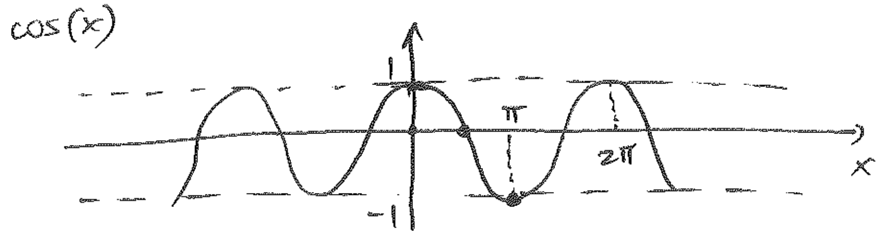
→ We want to model a function that looks like the data



→ what function has these features?

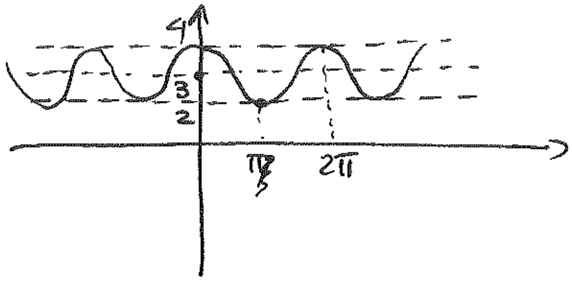
On the other hand here we have ~~a~~ functions which is ~~has~~ have the properties





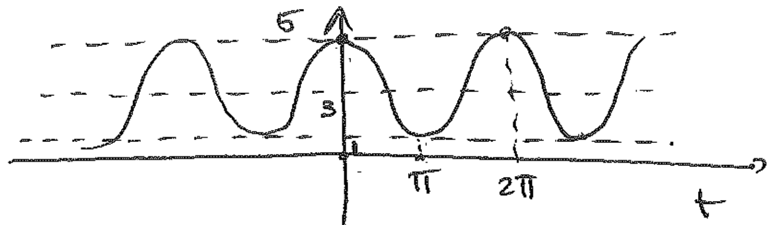
To model the tides function starting from the sine function, we therefore have to:

cos(t) + 3 → lifts cos by 3



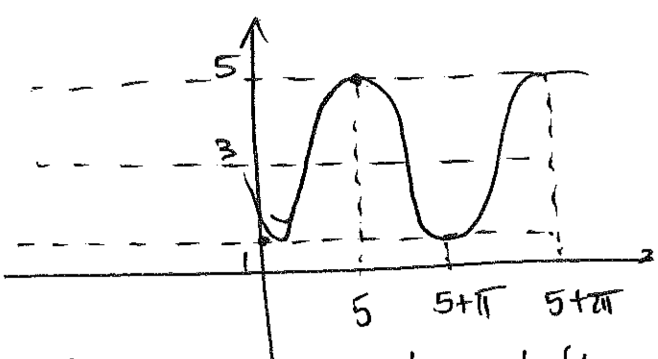
Adding 3 lifts the curve up by 3.

2cos(t) + 3



→ multiplying cos(t) by 2 increases the peak-to-trough difference

2cos(t-5) + 3



Shift it by 5 to get the maximum in the right place

→ to get it to have the right position for the minimum as well we need to multiply the independent variable by some coefficient.

→ 2cos(b (t-5)) + 3

Question: what should this coefficient be?

Using simple transformations on sine or cosine functions, we can therefore model many oscillatory functions. Let us now this how to do this in somewhat more generality.

6.4.2 Oscillatory functions

We now generalize what we saw in the previous section: the functions

$$f(x) = m + a\cos(b(x-c))$$
$$g(x) = m + a\sin(b(x-c))$$

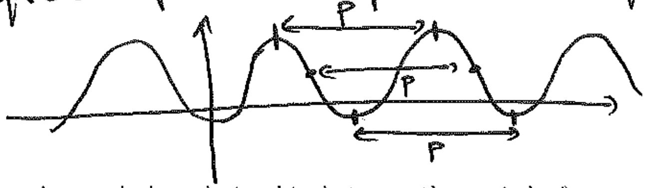
are oscillatory functions where:

- m is called the mean value
- a is called the amplitude (the difference between peak value & mean)
- c is the shift

These constants cause a vertical shift, a horizontal shift, and a vertical stretch of the function respectively. The interpretation of the constant b is a little less obvious, although we saw that it causes a horizontal stretching of the function. In fact, b is called the *frequency* of the oscillation, and it is related to the length of the pattern of oscillation, called the *period* of the oscillation.

DEFINITION: The constant b is called the frequency, and is related to the period of oscillation as $p = \frac{2\pi}{b}$

The period of oscillation is the length (or time) from peak to peak (or from troph-to-troph)



→ it the length of the basic pattern of oscillation.

To understand the relationship between the period of an oscillatory function p and the frequency b, note that

A periodic function satisfies $f(x) = f(x+p) = f(x-p)$ (with period p)

→ let's use this with $f(x)$ given above

$$f(x+p) = m + a\cos(b(x+p-c))$$
$$= m + a\cos(bx+bp-bc) = f(x)$$
$$= m + a\cos(bx-bc)$$

must be equal

$$\Rightarrow \quad \cos(\overbrace{bx-bc}^{\text{angle}}) = \cos(\overbrace{bx-bc}^{\text{angle}} + \overbrace{bp}^{2\pi})$$

$$\Rightarrow bp = 2\pi \Rightarrow \boxed{p = \frac{2\pi}{b}}$$

EXAMPLE: What are the mean, amplitude, period, frequency and shift of the following functions:

- $f(x) = 2 + 2\cos(2x + 2)$ $= 2 + 2\cos(2(x+1))$

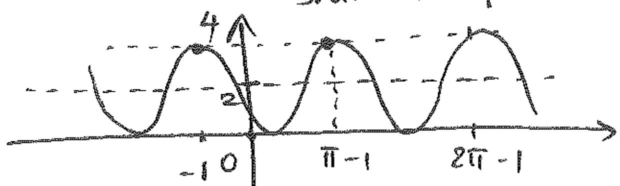
mean : 2

amplitude : 2

frequency : 2

shift : −1

period : $\frac{2\pi}{2} = \pi$



- $f(t) = \sin(2\pi t - 1)$ $= \sin\left(2\pi\left(t - \frac{1}{2\pi}\right)\right)$

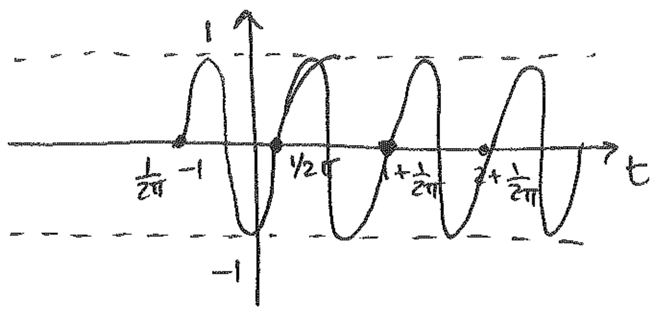
mean : 0

amplitude : 1

frequency : 2π

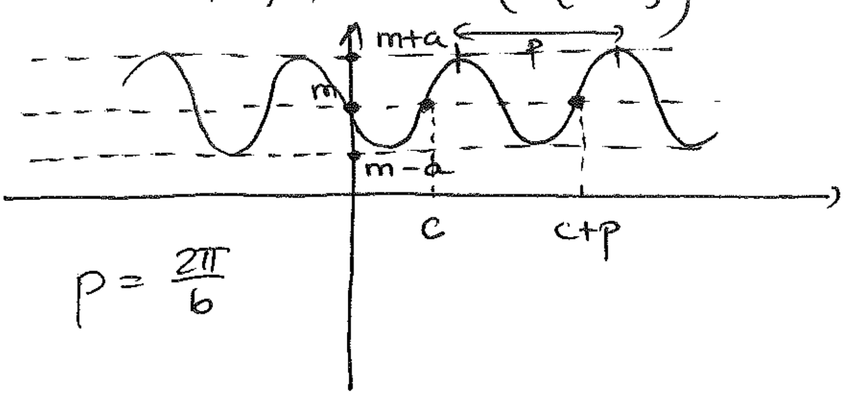
period : $\frac{2\pi}{2\pi} = 1$

phase shift : $\frac{1}{2\pi}$



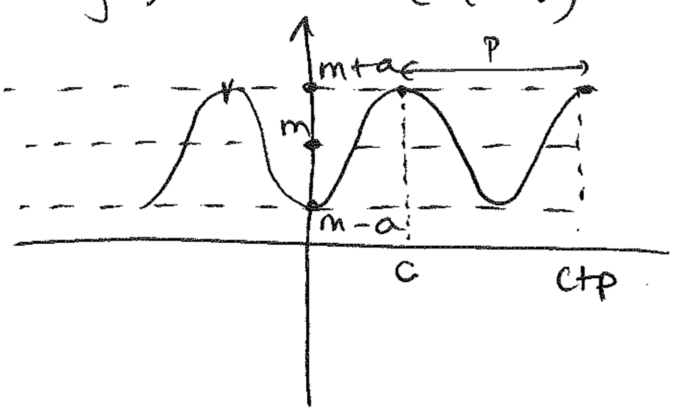
To summarize, here are the diagrams corresponding to the functions $f(x) = m + a\sin(b(x - c))$ and $g(x) = m + a\cos(b(x - c))$.

$f(x) = m + a\sin(b(x-c))$

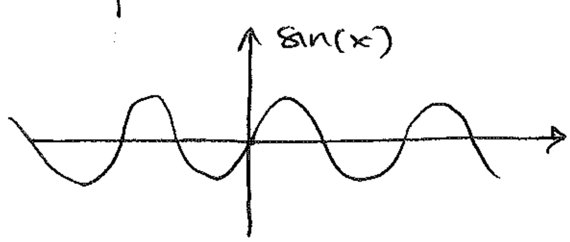


$p = \frac{2\pi}{b}$

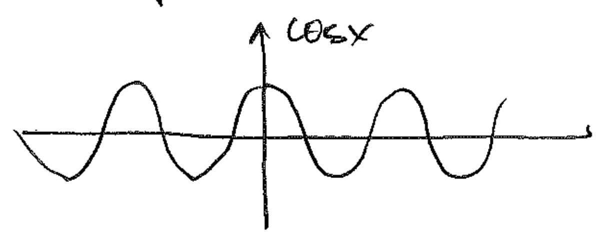
$g(x) = m + a\cos(b(x-c))$



(Reference function is



reference function is

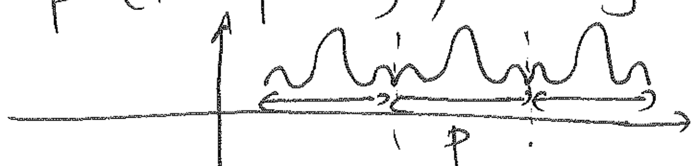


6.4.3 Periodic functions

The definition of a period can be extended to any function that has a pattern that repeats itself over a certain period, even if that function does not arise from a sine or a cosine function.

DEFINITION:

A periodic function is any function with the property that $f(x) = f(x+p)$ for a given number p (the period), for any x.



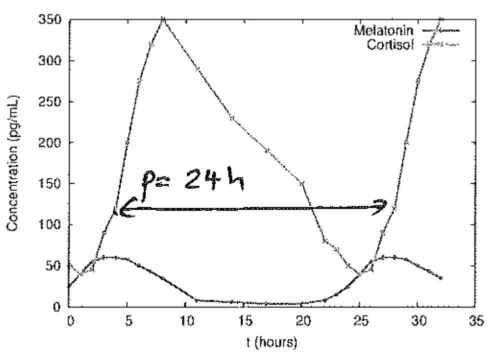
→ repeated pattern of length p.

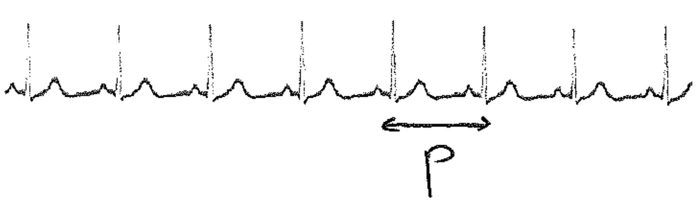
EXAMPLE

- The function $\tan(x)$ is periodic with period π. Indeed,

$$\tan(x) = \tan(x+\pi)$$

- Here are some real-life examples of periodic functions that are more complex than simple sinusoidal functions:





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