5.3 The natural exponential and the natural logarithm

As we have seen in the previous lecture, it is quite common to end up having to evaluate logarithms in all sorts of different bases. But if you look carefully on your calculator, you will only see two types of logs: the function **log**, which is actually the log in base 10, and the function **ln** which is another logarithm called the *natural logarithm*, that we will study now. As we will see in this lecture, it is possible to express the logarithm in base a in terms of the natural logarithm, and this is how quantities such as $\log_{1.01}(2)$ or $\log_2(3)$ can be calculated in practice.

Similarly, in scientific studies, it is very rare to use exponentials in many different bases. Instead, scientists tend to express any exponential function they come across in a new base called the *natural exponential.* We will now define both the natural exponential and the natural logarithm.

5.3.1 Definitions

DEFINITION:

The number e is a real number, with value approximately equal to:

The reason why this peculiar base is important in mathematics will be explored in more detail in Calculus, but just for a peek, note that the derivative of the function $f(x) = e^x$ (if you know what a derivative is) is *also* e^x, and this is the only function which is also its own derivative – and this is why it is so special.

Naturally, various function can be constructed from e^x:

DEFINITION:

PROPERTIES OF THE NATURAL LOGARITHM AND EXPONENTIAL: since these two functions are inverse of each other...

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-
-

5.3.2 Changing from base a to the natural exponential

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To change base from a base a exponential to the natural exponential (and vice versa, if needed):

The reason why this works is simple:

EXAMPLES:

- $2^x =$
- $\left(\frac{1}{4}\right)^x =$
- $2 \times 3^{-x} =$
- $4 \times 2^{-2x} =$

5.3.3 The growth and decay rates

Using the change of base formulas, we can ultimately express any growing or decaying exponential into one that is in base e:

In other words, our reference formula for exponentials can be rewritten as

where the plus or minus signs are used depending on whether the exponential is growing or decaying.

5.3.4 The doubling and halving constants.

Consider a growing exponential $f(x) = be^{rx}$.

Note that if x is a time, the doubling constant is called the doubling time. If x is a length, then the doubling constant is called the doubling length. Similarly, if we consider a decaying exponential $f(x) = be^{-rx}$ then

The doubling (or halving) constant only depends on the growth (or decay) rate r. To see this, let us solve for the doubling constant:

We see that the same formula applies for both growing and decaying exponentials! In addition, the formula is also independent of x, which means that:

EXAMPLE: What is the doubling time for a bank account with an interest rate of 3%?

5.3.5 Changing from base a to the natural logarithm

Just as we had a rule to change an exponential in base a to the natural exponential, there is also a rule to change a logarithm in base a to the natural logarithm:

The reason why this works is simple too:

EXAMPLES:

- $\log_2(x) =$
- $\log_{\frac{1}{4}} x =$

NOTE: This formula, when applied to a, yields the obvious relationship

If you are not sure of your change-of-base formula, this is a good way of double-checking that the formula you remember is the correct one.

This change of base is particularly useful because most calculators only provide $\ln(x)$ and not $\log_a(x)$. So, whenever you have to calculate $\log_a(x)$, you can use the formula to evaluate it using a normal calculator.

Example:

- How did I evaluate $\log_{1+r}(2)$?

- What is $\log_2(3)$?

- Solve the equation $2^x = 6$ and express the result as a natural logarithm.

- Show that for any a and b, the following is true: $\log_a(b)\log_b(a) = 1$.

5.3.6 Case study: *Exponential population growth*

In ecology, it is quite common to attempt to model the evolution of the size of the population of a given species under various model assumptions. The simplest such model is called the Exponential growth model (for reasons that will become clear shortly) and the premise of this model is to assume that the population of the species studied multiplies by a given factor over a known reproduction time, without any deaths or loss of reproductivity.

In this case study, we will look at the evolution of a population of rabbits in the wild. A pair of rabbits have on average 6 offsprings every 3 months. Based on this, what is the function that describes the evolution of a population of rabbits with initially one breeding pair, as a function of the number of months elapsed?

Let's now recast this function as a natural exponential:

We now see that the growth rate of the rabbit population is

We can also answer further questions such as how long does it take for the rabbit population to double?

How long does it take for this rabbit population to exceed the human population of the Earth (estimated at 7 billion)?

What could be wrong about this rabbit population model?