

2.2 Quadratic functions

Textbook Sections 2.3, 2.4 and 2.6

2.2.1 Case study: How to select the optimal pricing of Donnelly's chocolates?

Donnelly's in Santa Cruz (located by Bay and Mission) sell some of the finest chocolates in the US. This case study will show you how their company selects the pricing of their chocolates.

The cost of producing the chocolates is the following:

- *the rental of the facility costs \$200 per day,*
- *the salaries & benefits costs \$300/day*
- *the raw material for each chocolate costs \$1 per piece.*

What function $c(n)$ expresses the total production costs per day c as a function of the number of chocolates made n ?

What should one bear in mind when setting the price of the chocolates?

To find the optimal pricing for the chocolates, the owners decide to conduct a market analysis, i.e. a poll of their customers, to determine how many chocolates anyone would buy depending on the price of the chocolates. After interviewing all of their customers over the course of a week, they estimate that the number of chocolates n they would be able to sell per day as a function of their price p would be $n(p) = 1000 - 200p$. Let's graph this function, and interpret its meaning:

We now put all this information together, to find out how much money they would make each day m , as a function of the price of the chocolates p :

We can finally determine the optimal price for the chocolates, using Wolfram Alpha to graph the function $m(p)$.

In this case study, we ended up solving our problem graphically. However, with the right mathematical tools, we can also do it without graphs. Let's learn more about quadratics.

2.2.2 Mathematical corner: Properties of quadratic functions

A quadratic function is a special type of polynomial function. The general expression for a quadratic function is

The graph of all quadratic functions is called a *parabola*. The exact shape and position of the parabola depends on the coefficients of the quadratic. Different cases can arise:

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OPENING UP OR DOWN. Whether a parabola opens “up” or “down” can very easily be determined simply by inspection of the quadratic term ax^2 in the function.

Let’s consider two examples of quadratic functions:

- $f(x) = 3x^2 - 2x - 1$

- $g(x) = -2x^2 + x + 1$

and graph them on Wolfram Alpha. We also compare their graphs with those of the functions $3x^2$ and $-2x^2$. We notice that:

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This is in fact true of all quadratics!

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BEHAVIOR NEAR THE y -AXIS What the parabola looks like near the y -axis (i.e. when x is close to 0) can *also* very easily be determined simply by inspection of the quadratic function, but this time, of the $bx+c$ bit.

Let's consider the functions $f(x)$ and $g(x)$ again, but this time zoom in the graph near $x = 0$. We also compare their graphs with those of the functions $-2x - 1$ and $x + 1$. We notice that:

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Again, this is true for every quadratic!

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VERTEX OF A PARABOLA AND VERTEX FORM

We already saw in the previous case study an example where we were interested in finding out what the coordinates of the vertex are (in that case, where the maximum of the graph was). We can now check analytically our graphical result. Indeed,

Here are further examples. In each case, let's use the vertex coordinates formula to find the position of the vertex, then use the vertex form of the quadratic to sketch the graph of the function using basic transformations.

- $f(x) = x^2 - 1$

- $f(x) = -3x^2 + 4$

- $f(x) = x^2 - 4x + 4$

- $f(x) = x^2 + 6x + 10$

- $f(x) = -5x^2 + 30x - 46$

2.2.3 Case study: *How to select the optimal pricing of Donnelly's chocolates?*

In the previous lecture, we learned how varying the selling price of Donnelly's chocolates affects the expected number of chocolates sold in any given day, and therefore the net profit that the company can make. We then selected the pricing to maximize profit. Alternatively, the company may decide to sell the chocolates a little below this price to attract new clients. How low a price can they sell their chocolates without losing any money? Conversely, how high a price could they try to sell them without losing money?

2.2.4 Mathematical corner: Roots of quadratics

DEFINITION:

As we saw in the last lecture, some parabolas cross the x -axis, and some do not. Graphically, we know that there are 3 possible cases:

Since a parabola is the graph of a quadratic function, this means that some quadratic functions $f(x) = ax^2 + bx + c$ have 2 roots (i.e. 2 x -intercepts), some have one root (i.e. 1 x -intercept), and some do not have any. In other words,

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FACTORED FORM:

The symbol \Leftrightarrow implies that there is a strict equivalence relationship between the two statements “ $f(x)$ can be factored” and “ $f(x)$ has roots”: the first implies the second, and conversely, the second implies the first. In fact, it is quite easy to check that, in both cases, the second statement implies the first. Indeed, let us look for the solutions of $f(x) = 0$ from its factored form.

EXAMPLES

- $f(x) = 3(x - 1)(x + 2)$

- $f(x) = \frac{1}{2}(x - 4)^2$

The interesting thing about equivalence statements in logic is that if you have one, then you also have equivalence of the opposites:

EXAMPLES

- $f(x) = x^2 + 4$

- $f(x) = x^2 + 2x + 4$

FACTORING QUADRATICS. Based on what we just saw, it would be nice to have simple tricks to

- tell us when a quadratic has roots or not,
- or equivalently, factor the quadratic if it can be factored.

As it turns out, there are a few types of quadratics that can very easily be factored:

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EXAMPLES

- $f(x) = x^2 - 2$

- $f(x) = 2x^2 - 3$

- $f(x) = x^2 + 6x + 9$

- $f(x) = 2x^2 + 4\sqrt{5}x + 10$

- $f(x) = -x^2 + 10x - 25$

On the other hand, not every quadratic is in one of these three “ideal forms”. What can we do if it isn’t? As it turns out, another nice trick exists in that case, and is called “The quadratic formula”.

THE QUADRATIC FORMULA. Given the quadratic $ax^2 + bx + c$,

- Calculate the discriminant $D = b^2 - 4ac$
- If $D < 0$ there are no solutions to the equation $ax^2 + bx + c = 0$, and the quadratic cannot be factored.
- If $D = 0$ there is one solution to the equation $ax^2 + bx + c = 0$, $x = -\frac{b}{2a}$ and the quadratic can be factored as
- If $D > 0$ there are two solutions to the equation $ax^2 + bx + c = 0$, which are
and the quadratic can be factored as

NOTE: The vertex of a parabola is always half-way between the roots! Indeed,

We can now also use this technique to solve the equation associated with our case study earlier, in a different way:

Finally, it is worth noting that this method can also help solve a few higher-order equations that can be reduced to a quadratic, as in these examples:

- What are the solutions (if any) to the equation $f(x) = x^6 - 3x^3 - 9 = 0$?

- What are the solutions (if any) to the equation $f(x) = x^4 - 2x^2 - 3 = 0$?

GRAPHING QUADRATICS. Let's now recap everything we know about the graphs of quadratic functions based on their mathematical expression. Given $f(x) = ax^2 + bx + c$,

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- Given $D = b^2 - 4ac$:

Based on all this information (much of which is actually redundant) we can easily graph the parabola.

EXAMPLE 1: $f(x) = 2x^2 - 3x + 1$

EXAMPLE 2: $f(x) = -x^2 - 4x - 4$

EXAMPLE 3: $f(x) = -x^2 + x - 1$