Practice Final 2, AMS 3

Name: ________________________________

Calculators are not allowed.

Read all the questions before you start working on any of them. Start with the ones you are most comfortable with, and continue with the other ones later. Always double-check your answers.

Relax, and do your best!

Problem 1: Short questions. [40 points]

1. What is the equation of the line parallel to $y = -2x$ which goes through the point $(1, 2)$?

Slope is -2

so $y - 2 = -2(x-1)$ $\quad y = 2 - 2x + 2 = 4 - 2x$

Given the functions $f(x) = e^{-2x}$ and $g(x) = \ln(x)$

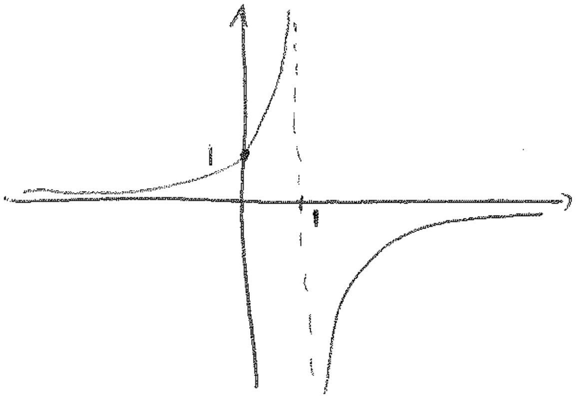
2. What is the domain of f? $x \in \mathbb{R}$ $(\mathbb{R}$ $(-\infty, +\infty))$

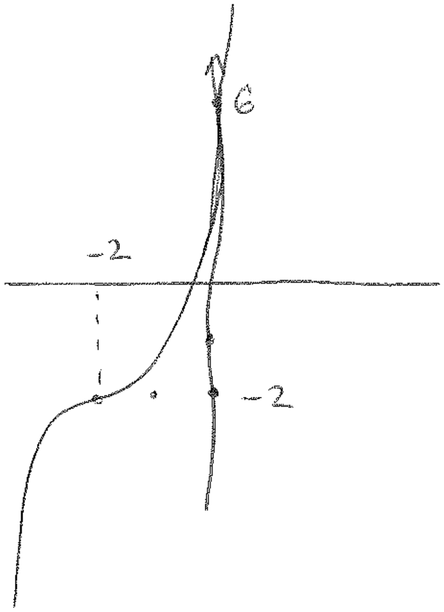
3. Simplify $f \circ g(x)$? $e^{-2\ln x} = e^{\ln(x^{-2})} = x^{-2} = 1/x^2$

4. Simplify $g \circ f(x)$? $\ln(e^{-2x}) = -2x$

5. Is the function $f(x) = x^3 - \frac{1}{x}$ odd, even, or neither? odd

6.,7. Sketch the functions $f(x) = -\frac{1}{x-1}$ and $g(x) = (x+2)^3 - 2$





8. Factor the expression $3x^2 + x - 1$ (if it can be factored; if not say so):

$$D = (1)^2 - 4(3)(-1) = 1 + 12 = 13 \qquad x_{1,2} = \frac{-1 \pm \sqrt{13}}{6}$$

$$\text{so} \quad 3x^2 + x - 1 = 3(x - x_1)(x - x_2) = 3\left(x - \frac{-1+\sqrt{13}}{6}\right)\left(x - \frac{-1-\sqrt{13}}{6}\right)$$

Given the parabola $y = 2x^2 + x - 5$:

9. What is the x-coordinate of the vertex? $-\frac{1}{2(2)} = -1/4$

10. Does it open up or down? up

11. What is the y-intercept? -5

12. What are the x-intercepts? $D = (1)^2 - 4(2)(-5) = 1 + 40 = 41$

$$x_{1,2} = \frac{-1 \pm \sqrt{41}}{2(2)} = \frac{-1 \pm \sqrt{41}}{4}$$

13. Draw a signs table for the function $f(x) = (x^2 - 4x^4)(x^2 - 6x + 9)$ $= x^2(1 - 4x^2)(x-3)^2$

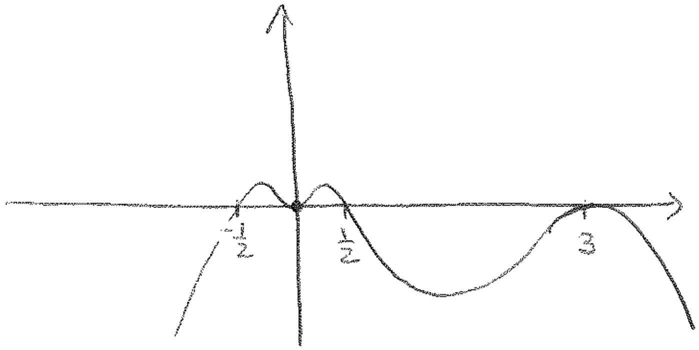
$= x^2(1-2x)(1+2x)(x-3)^2$

↓ root at $x = \frac{1}{2}$

↓ root at $x = -\frac{1}{2}$

		$-1/2$		0		$1/2$		3	
x^2	+		+	0	+		+		+
$1-2x$	+		+		+	0	−		−
$1+2x$	−	0	+		+		+		+
$(x-3)^2$	+		+		+		+	0	+
	−	0	+	0	+	0	−	0	−

14. Sketch the function $f(x)$ of question 13.



15. Given the function $f(x)$ of question 13, what is the domain of $\ln(f(x))$?

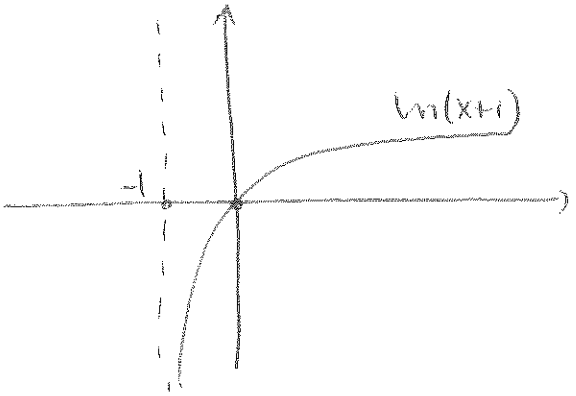
$$\mathcal{D} = \left(-\tfrac{1}{2}, 0\right) \cup \left(0, \tfrac{1}{2}\right)$$

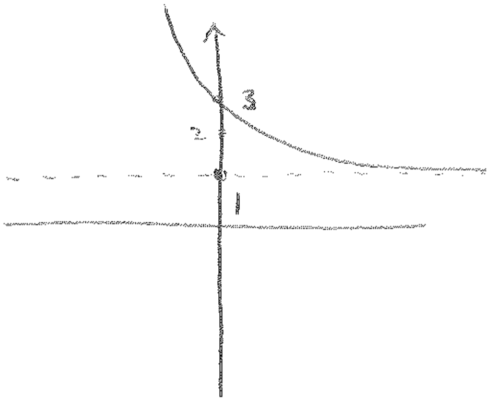
16. Simplify $f(x) = \frac{9^x 3^{-x}}{3^{3x} 27^{-x}}$ $= \frac{3^{2x}\ 3^{-x}}{3^{3x}\ 3^{-3x}} = \frac{3^x}{3^0} = 3^x$

17. Simplify $\log_4(16x^3) + 2\log_4(1/x)$

$\log_4(16x^3) - \log_4(x^2) = \log_4\left(\frac{16x^3}{x^2}\right) = \log_4(16x) = 2 + \log_4(x)$

18. 19. Sketch the functions $\ln(x+1)$ and $2e^{-x} + 1$.





20. Expand the function $f(x) = \ln\left[\frac{2e^x(x-1)^3(x-2)}{(4x-1)^5(x+3)}\right]$ into a sum (or difference) of logarithms.

$= \ln 2 + \ln e^x + \ln((x-1)^3) + \ln(x-2) - \ln((4x-1)^5) - \ln(x+3)$

$= \ln 2 + x + 3\ln(x-1) + \ln(x-2) - 5\ln(4x-1) - \ln(x+3)$

21. Simplify $\log_3(e^{2x})$ $= \frac{\ln(e^{2x})}{\ln 3} = \frac{2x}{\ln 3}$

22. Write $20\left(\frac{1}{2}\right)^{3x}$ as an exponential in base e. $= 20e^{3x\ln(\frac{1}{2})} = 20e^{-3x\ln 2}$

23. Solve the equation $x^4 + 3x^2 + 2 = 0$ for x Let $u = x^2$ then $u^2 + 3u + 2 = 0$

$\rightarrow (u+1)(u+2) = 0 \rightarrow u = -1$ or $u = -2$

or $\begin{cases} \rightarrow x^2 = -1 & \text{(no solution)} \\ \rightarrow x^2 = -2 & \text{(no solution)} \end{cases}$

→ Thus equation has no solutions.

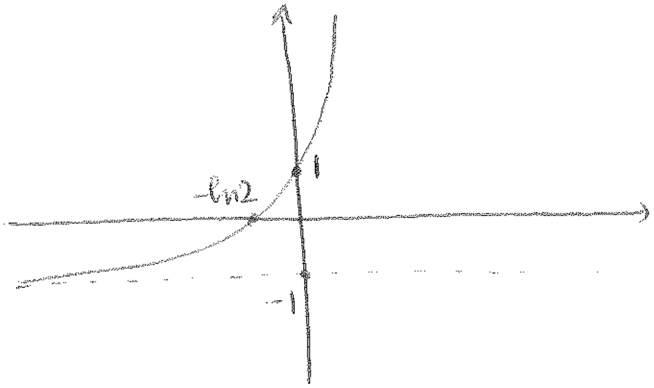
24.,25. Calculate the x- and y-intercepts of the function $f(x) = 2e^x - 1$, then sketch the function. Hint: $\ln(2) \simeq 0.7$.

$f(0) = 2e^0 - 1 = 1$

$2e^x - 1 = 0$

$e^x = \frac{1}{2}$

$x = \ln\left(\frac{1}{2}\right) = -\ln 2$



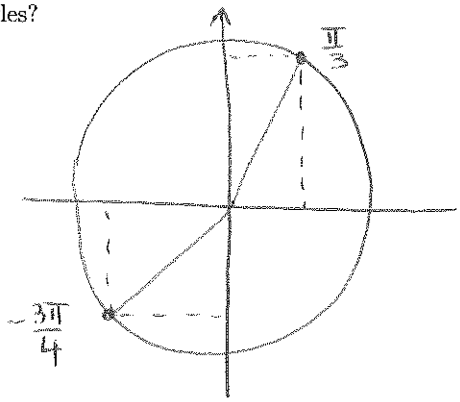
26. Solve the equation $4^x = 5^{1/x}$

$\ln(4^x) = \ln(5^{1/x}) \rightarrow x\ln 4 = \frac{1}{x}\ln 5 \rightarrow x^2 = \frac{\ln 5}{\ln 4}$

$\rightarrow x = \pm\sqrt{\frac{\ln 5}{\ln 4}}$

27. $\log_4(2x^3) = 3\log_4(2x)$: TRUE/[✓ FALSE]

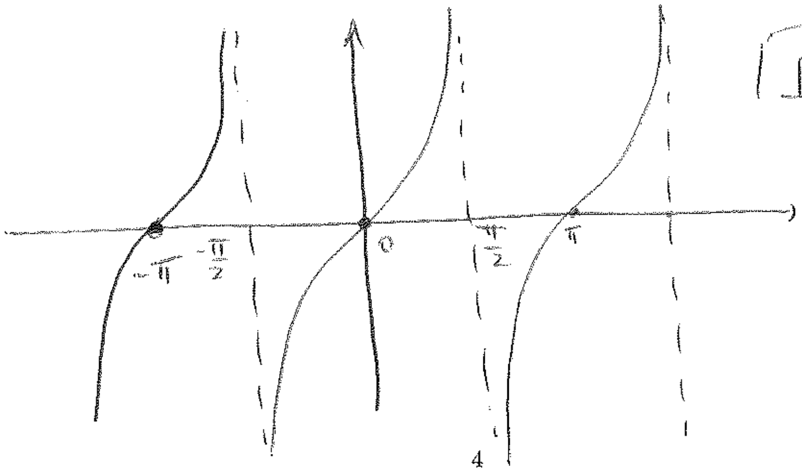
28.,29.,30.,31. Draw a unit circle, and place the angles $\pi/3$, and $-3\pi/4$. What are the tangents of these angles?



$\tan\frac{\pi}{3} = \frac{\sin \pi/3}{\cos \pi/3} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$

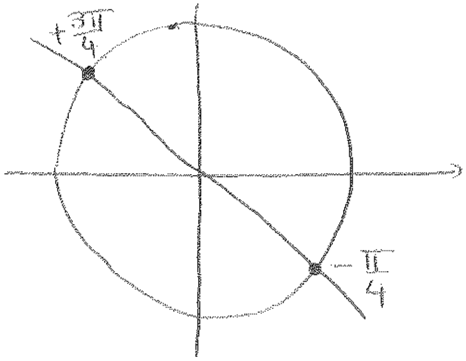
$\tan\left(-\frac{3\pi}{4}\right) = \frac{\sin\left(-\frac{3\pi}{4}\right)}{\cos\left(-\frac{3\pi}{4}\right)} = \frac{-\sqrt{2}/2}{-\sqrt{2}/2} = 1$

32. 33. Sketch the function $\tan(x)$. What is the period of the function $\tan(x)$?



$\boxed{\text{period is } \pi}$

34.,35. Draw all the points on the unit circle where $\sin(x) = -\cos(x)$, and solve this equation for x.



$$\rightarrow \quad x = -\frac{\pi}{4} + 2k\pi$$

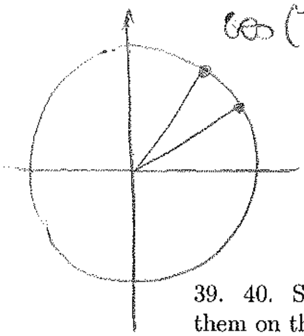
$$x = \frac{3\pi}{4} + 2k\pi$$

36. Simplify $3\cos^2(x) + 2\sin^2(x) - 1$ using the Pythagorean formula

$$= \cos^2 x + 2\cos^2 x + 2\sin^2 x \quad -1$$

$$= \cos^2 x \quad + 2 - 1 = \cos^2 x + 1$$

37. 38. Use the formula $\cos(a - b) = \cos a \cos b + \sin a \sin b$ to calculate $\cos(\pi/12)$.
Hint: write $\pi/12$ as $\pi/4 - \pi/6$.

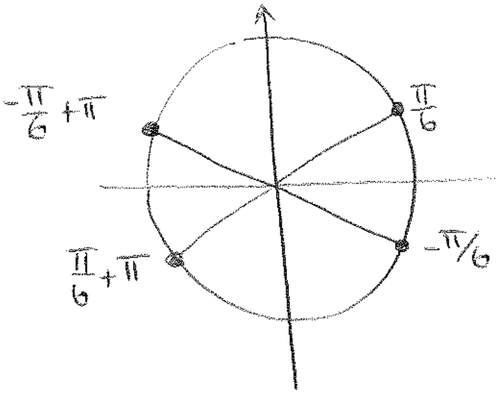


$$\cos\left(\frac{\pi}{12}\right) = \cos\left(\frac{\pi}{4} - \frac{\pi}{6}\right) = \cos\left(\frac{\pi}{4}\right) \cos\left(\frac{\pi}{6}\right) + \sin\left(\frac{\pi}{4}\right) \sin\left(\frac{\pi}{6}\right)$$

$$= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{2}}{2}\left(\frac{\sqrt{3}}{2} + \frac{1}{2}\right) = \frac{\sqrt{2}}{4}(1 + \sqrt{3})$$

39. 40. Solve the equation $\cos(2x) = \frac{1}{2}$ for x. Make sure you write all possible solutions, and draw them on the unit circle.

let $u = 2x$ then $\cos u = \frac{1}{2} \rightarrow u = \frac{\pi}{3} + 2k\pi$ or $-\frac{\pi}{3} + 2k\pi$



so $x = \frac{1}{2} u = \begin{cases} \frac{\pi}{6} + k\pi \\ -\frac{\pi}{6} + k\pi \end{cases}$

PROBLEM 2: POLYNOMIAL .[15 POINTS] Consider the function $f(x) = (2x^2 - 3x)(4x^2 + 7x + 3)$.

(a) Factor and simplify this expression.

$$f(x) = x(2x-3)(4x^2+7x+3)$$

$$D = 49 - 4(4)(3) = 49 - 48 = 1$$

so $x_{1,2} = \frac{-7 \pm \sqrt{1}}{8} = \frac{-7 \pm 1}{8} = \begin{cases} -1 \\ -\frac{6}{8} \end{cases}$

$$= x(2x-3)4(x+1)(x+\tfrac{3}{4})$$

$$= x(2x-3)(x+1)(4x+3)$$

(b) What is the y–intercept? 0

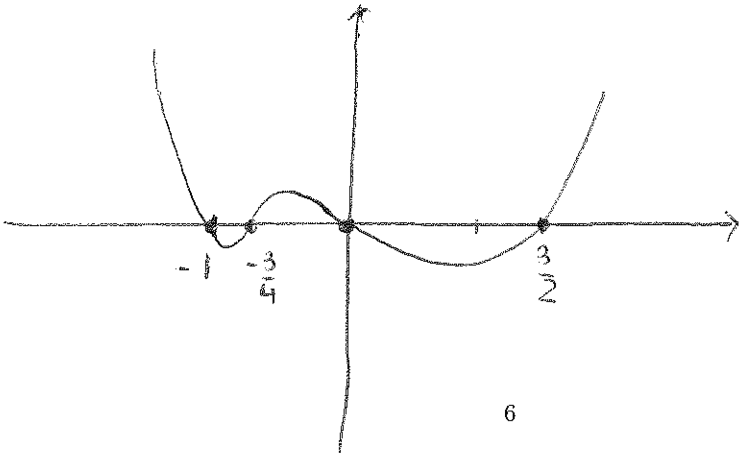
(c) What are the x–intercepts? $0, \frac{3}{2}, -1, -\frac{3}{4}$

(d) What is the behavior of $f(x)$ as x goes to $+\infty$ and $-\infty$? looks like $8x^4$ so goes to $+\infty$ on both sides

(e) Draw a signs table for $f(x)$

		-1		-3/4		0		3/2	
x	−		−		−	0	+		+
2x−3	−		−		−		−	0	+
x+1	−	0	+		+		+		+
4x+3	−		−	0	+		+		+
	+	0	−	0	+	0	−	0	+

(f) Using this information, sketch $f(x)$, making sure to annotate your graph with all the important points and features.



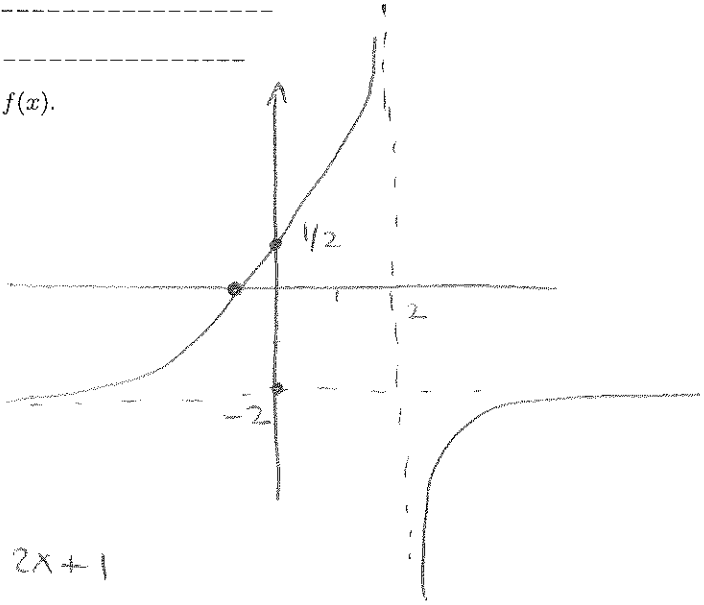
PROBLEM 3. RATIONAL FUNCTIONS AND INVERSES [15 POINTS] Consider the function $f(x) = \frac{2x+1}{-x+2}$

(a) What are:

- The $x-$intercept: $-1/2$
- The $y-$intercept: $1/2$
- The horizontal asymptote: -2
- The vertical asymptote: 2

(b) Draw a signs table for $f(x)$ and sketch the graph of $f(x)$.

		$-1/2$		2	
$2x+1$	$-$	0	$+$		$+$
$-x+2$	$+$		$+$	∞	$-$
	$-$	0	$+$	∞	$-$



(c) What is $f^{-1}(x)$?

$$y = \frac{2x+1}{-x+2} \to (-x+2)y = 2x+1$$

$$\to -xy+2y = 2x+1 \to -xy-2x = 1-2y \to x(-y-2) = 1-2y$$

$$\text{So} \quad x = \frac{1-2y}{-y-2} = \frac{2y-1}{y+2}$$

$$f^{-1}(x) = \frac{2x-1}{x+2}$$

(d) Verify that $f[f^{-1}(x)] = x$.

$$\frac{2\left(\frac{2x-1}{x+2}\right)+1}{-\left(\frac{2x-1}{x+2}\right)+2} = \frac{\frac{2(2x-1)+x+2}{x+2}}{\frac{-(2x-1)+2(x+2)}{x+2}} = \frac{4x\cancel{-2}+x\cancel{+2}}{\cancel{x+2}} \; \frac{\cancel{x+2}}{\cancel{-2x}+1\cancel{+2x}+4}$$

$$= \frac{5x}{5} = x \checkmark$$

APPLIED PROBLEM 1. TRIGONOMETRIC FUNCTIONS. [15 POINTS]

The price of a 1/2 pound basket of strawberries in Oregon as a function of time is given by the function $p(t) = 8 + 4\cos\left[\frac{\pi}{6}(t-1)\right]$ where t is measured in the number of months since January first (January 1st being $t = 0$), and p is in dollars.

(a) What is the mean price of strawberries over the whole year? 8 dollars

(b) What is the price of strawberries on January first? (Hint: $\sqrt{3} \simeq 1.7$)

$$p(0) = 8 + 4\cos\left[\frac{\pi}{6}(0-1)\right] = 8 + 4\cos\left(-\frac{\pi}{6}\right) = 8 + 4\left(\frac{\sqrt{3}}{2}\right) = 8 + 2\times 1.7 = 11.4$$

(c) What is the highest possible price the strawberries reach, and at what month does this occur?

- Price is highest at maximum value of oscillation, price is $8+4 = 12$
- This happens when $\cos\left(\frac{\pi}{6}(t-1)\right) = 1$ so when

$$\frac{\pi}{6}(t-1) = 0, (2\pi) \ldots \qquad t-1 = 0 \to t = 1$$

→ February.

(d) What is the lowest possible price the strawberries reach, and at what month does this occur?

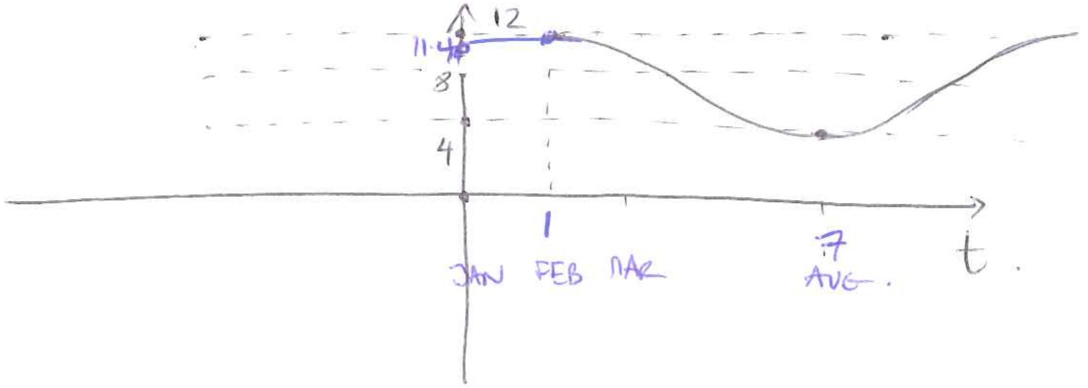
- Price is lowest at minimum of oscillation, price is $8-4=4$
- This happens when $\cos\left(\frac{\pi}{6}(t-1)\right) = -1$ so when

$$\frac{\pi}{6}(t-1) = +\pi \quad (+2\pi k)$$

$$\Rightarrow t-1 = \frac{6}{\cancel{\pi}}\cdot\cancel{\pi} \to t = 6+1 = 7 \to \text{August}$$

Jan 0 Feb 1 March 2 Apr 3 May 4 June 5 July 6 August 7 Sept 8.

(e) Sketch the function $p(t)$, making sure to indicate all the important features and points.



APPLIED PROBLEM 2. [15 POINTS]

An alien species has landed on Earth in 2030, and has decided to exterminate the human race. Every 2 weeks, they kill off one third of the remaining population. When they first land, there are 9 billion human beings on Earth.

(a) How many survivors are left

- 2 week after their landing? $\frac{2}{3}\,9\,10^9 = 6\,10^9$ (6 billion)
- 4 weeks after their landing? $\frac{2}{3}\,6\,10^9 = 4\,10^9$ (4 billion)
- 6 weeks after their landing? $\frac{2}{3}\,4\,10^9 = \frac{8}{3}\,10^9$ (2.666 billion)

(b) Which of the following functions would best describe the number of survivors S after n weeks?

- $S(n) = 9 \times 10^9 \times \left(\frac{1}{3}\right)^n$
- $S(n) = 9 \times 10^9 \times \left(\frac{2}{3}\right)^n$
- $S(n) = 9 \times 10^9 \times \left(\frac{1}{3}\right)^n$
- [✓ $S(n) = 9 \times 10^9 \times \left(\frac{2}{3}\right)^{n/2}$]

(c) Write the selected function $S(n)$ in the form of an exponential in base e: $9\,10^9\, e^{\frac{n}{2}\ln\left(\frac{2}{3}\right)}$

(d) What is the halving time of the function $S(n)$? (Note: you can either answer this question by remembering the formula for the doubling time, or by solving a particular equation).

Solution 1: $\text{time} = \frac{\ln 2}{r} = \frac{\ln 2}{-\frac{1}{2}\ln\left(\frac{2}{3}\right)} = \frac{2\ln 2}{-\ln\left(\frac{2}{3}\right)} = \frac{2\ln\left(\frac{1}{2}\right)}{\ln\left(\frac{2}{3}\right)}$

Solution 2: Solve $4.5\,10^9 = 9\,10^9\left(\frac{2}{3}\right)^{\frac{n}{2}}$

$\to \frac{1}{2} = \left(\frac{2}{3}\right)^{\frac{n}{2}} \to \ln\left(\frac{1}{2}\right) = \frac{n}{2}\ln\left(\frac{2}{3}\right) \to n = \frac{2\ln\left(\frac{1}{2}\right)}{\ln\left(\frac{2}{3}\right)}$

(e) How long does it take until a single human is left? (You do not need to simplify nor evaluate your answer. The actual answer is about 2 years).

$$1 = 9\,10^9\left(\frac{2}{3}\right)^{\frac{n}{2}}$$

$$\frac{1}{9\,10^9} = \left(\frac{2}{3}\right)^{\frac{n}{2}} \to \ln\left(\frac{1}{9\,10^9}\right) = \frac{n}{2}\ln\left(\frac{2}{3}\right)$$

$$\to n = \frac{2\ln\left(\frac{1}{9\,10^9}\right)}{\ln\left(\frac{2}{3}\right)} = \frac{-2\ln\left(9\,10^9\right)}{\ln\left(\frac{2}{3}\right)}$$

$n \simeq 112$ weeks