Practice Final 1, AMS 3

Name: ________________________________

Calculators are not allowed.

Read all the questions before you start working on any of them. Start with the ones you are most comfortable with, and continue with the other ones later. Always double-check your answers.

Relax, and do your best!

PROBLEM 1: SHORT QUESTIONS. [40 POINTS]

Passes through

1. What linear function would satisfy $f(3) = 2$ and $f(1) = 1$? A(3,2) B(1,1)

$$m = \frac{2-1}{3-1} = \frac{1}{2}$$

$$(y-2) = \frac{1}{2}(x-3) \rightarrow y = 2 + \frac{1}{2}x - \frac{3}{2} = \boxed{\frac{1}{2}x + \frac{1}{2} = f(x)}$$

Check: $f(3) = \frac{3}{2} + \frac{1}{2} = 2$ ✓ $f(1) = \frac{1}{2} + \frac{1}{2} = 1$ ✓

Given the functions $f(x) = e^x$ and $g(x) = \ln(x-2)$

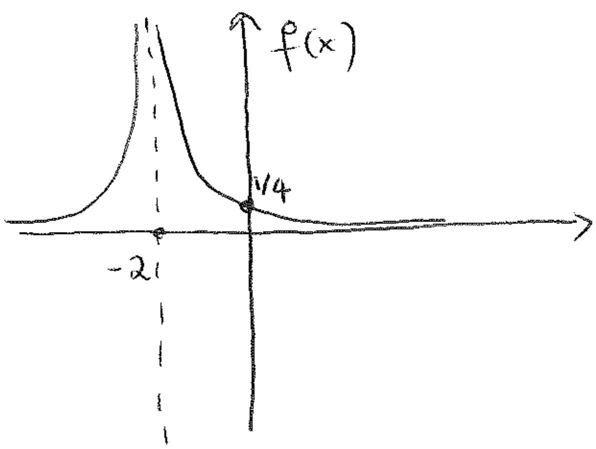
2. What is the domain of g? $x > 2$ $(2, +\infty)$

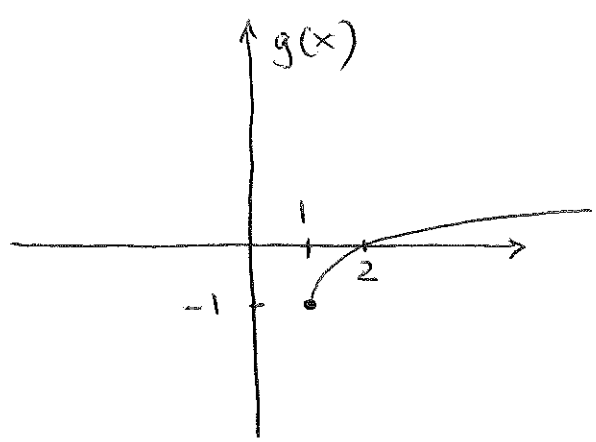
3. What is $f \circ g(x)$? $e^{\ln(x-2)} = x - 2$

4. What is $g \circ f(x)$? $\ln(e^x - 2)$ (cannot be simplified)

5. Given the function $f(x) = \ln(\sqrt{x})$ for $x > 0$, what is $f^{-1}[f(x^4+1)]$? $x^4 + 1$

6.,7. Sketch the functions $f(x) = \frac{1}{(x+2)^2}$ and $g(x) = \sqrt{x-1} - 1$





8. Factor the expression $2x^2 - 2x + 1$ (if it can be factored; if not, say so):

$D = (-2)^2 - 4(2)(1) = 4 - 8 = -4$

$\rightarrow$ cannot be factored

Given the parabola $y = x^2 - 2x - 6$:

9. What are the coordinates of the vertex? $-\frac{(-2)}{2(1)} = \frac{2}{2} = 1$

10. Does it open up or down? Up

11. What is the y-intercept? -6

12. What are the x-intercepts? $D = (-2)^2 - 4(1)(-6) = 4 + 24 = 28$

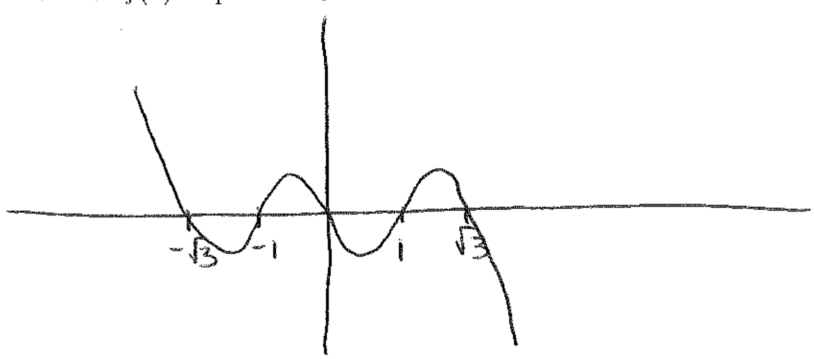
$$x_{1,2} = \frac{-(-2) \pm \sqrt{28}}{2(1)} = \underbrace{\frac{2 \pm \sqrt{28}}{2}}_{\text{ok here}} = 1 \pm \sqrt{7}$$

13. Draw a signs table for the function $f(x) = (x^3 - x)(3 - x^2)$ $= x(x^2-1)(3-x^2)$

$= x(x-1)(x+1)(\sqrt{3}-x)(\sqrt{3}+x)$

		$-\sqrt{3}$		-1		0		1		$\sqrt{3}$	
x	$-$		$-$		$-$	0	$+$		$+$		$+$
$x-1$	$-$		$-$		$-$		$-$	0	$+$		$+$
$x+1$	$-$		$-$	0	$+$		$+$		$+$		$+$
$\sqrt{3}-x$	$+$		$+$		$+$		$+$		$+$	0	$-$
$\sqrt{3}+x$	$-$	0	$+$		$+$		$+$		$+$		$+$
	$+$	0	$-$	0	$+$	0	$-$	0	$+$	0	$-$

14. Sketch the function $f(x)$ of question 13.



15. Given the function $f(x)$ of question 13, what is the domain of $\ln(f(x))$? Need $f(x) > 0$ so

$\mathcal{D} = (-\infty, -\sqrt{3}) \cup (-1, 0) \cup (1, \sqrt{3})$

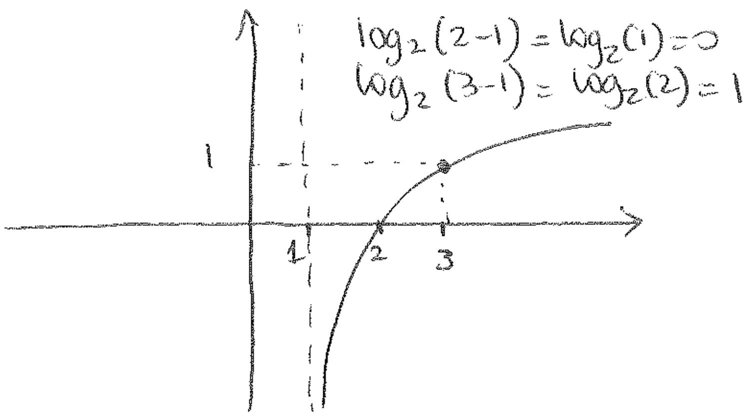
16. Simplify $f(x) = \frac{2^x 4^{-2x}}{8^{3x} 2^{-x}}$ $= \dfrac{2^x \; 2^{-4x}}{2^{9x} \; 2^{-x}} = \dfrac{2^{-3x}}{2^{8x}} = 2^{-11x}$

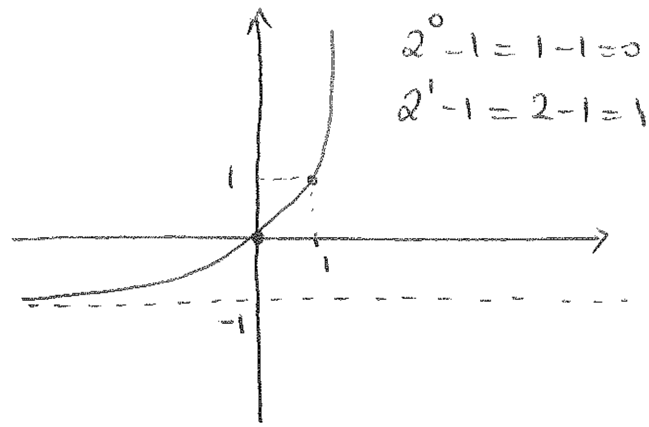
because $4^{-2x} = (2^2)^{-2x} = 2^{-4x}$ $\quad 8^{3x} = (2^3)^{3x} = 2^{9x}$

17. Simplify $\log_4(2x^2) - 2\log_4(x)$

$= \log_4(2x^2) - \log_4(x^2) = \log_4\left(\frac{2\cancel{x^2}}{\cancel{x^2}}\right) = \log_4(2) = \frac{1}{2}$

18. 19. Sketch the functions $\log_2(x-1)$ and $2^x - 1$, making sure to indicate the y-intercept and other important points / lines.





20. Expand the function $f(x) = \ln\left[\frac{2x^2(x-1)(x+1)}{4x^3}\right]$ into a sum (or difference) of logarithms.

$f(x) = \ln(2x^2) + \ln(x-1) + \ln(x+1) - \ln(4x^3)$

$= \ln 2 + 2\ln x + \ln(x-1) + \ln(x+1) - \ln 4 - 3\ln x$

$= \ln\left(\frac{1}{2}\right) - \ln x + \ln(x-1) + \ln(x+1)$ b.c. $\ln 2 - \ln 4 = \ln\left(\frac{2}{4}\right) = \ln\frac{1}{2}$

21. Simplify $\log_2(e^x)$ $= \dfrac{\ln(e^x)}{\ln 2} = \dfrac{x}{\ln 2}$

22. What is the inverse of the function $f(x) = 4x^{-1/4}$?

$y = 4x^{-1/4} \rightarrow \frac{y}{4} = x^{-1/4} \Rightarrow \left(\frac{y}{4}\right)^{-4} = x$ so

$f^{-1}(x) = \left(\frac{x}{4}\right)^{-4}$

23. What is the inverse of the function $f(x) = 2^{x+1} - 3$

$y = 2^{x+1} - 3 \rightarrow (y+3) = 2^{x+1} \rightarrow \ln(y+3) = \ln(2^{x+1})$

$\rightarrow (x+1)\ln 2 = \ln(y+3) \rightarrow x+1 = \dfrac{\ln(y+3)}{\ln 2}$

$\rightarrow x = \dfrac{\ln(y+3)}{\ln 2} - 1$

$f^{-1}(x) = \dfrac{\ln(x+3)}{\ln 2} - 1$ or $\log_2(x+3) - 1$

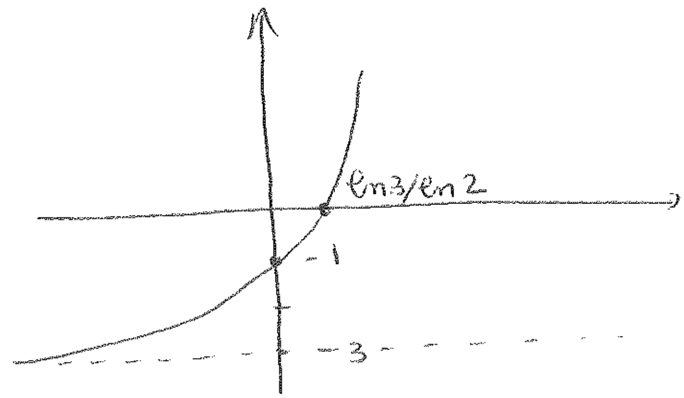
24.,25. Calculate the x- and y-intercepts of the function $f(x) = 2^x - 3$, then sketch the function.

$f(0) = 2^0 - 3 = -1$

$2^x - 3 = 0 \to 2^x = 3$

$\to x = \frac{\ln 3}{\ln 2}$

or $x = \log_2(3)$



26. Solve the equation $2^x = 3^{x-1}$

$\ln(2^x) = \ln(3^{x-1})$

$\to x\ln 2 = (x-1)\ln 3 \to x\ln 2 = x\ln 3 - \ln 3$

$\to x(\ln 2 - \ln 3) = -\ln 3 \to x = \frac{-\ln 3}{\ln 2 - \ln 3}$

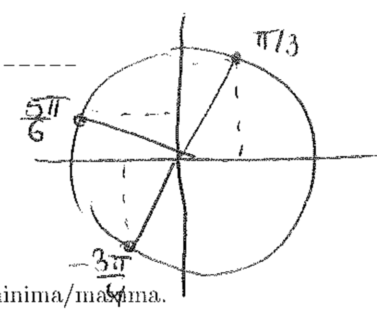
27. $\log_4(x^3) = \frac{3}{\ln(4)}\ln(x)$: TRUE/FALSE

28. Write $2^{x/a}$ as a natural exponential. $e^{\frac{x}{a}\ln 2}$

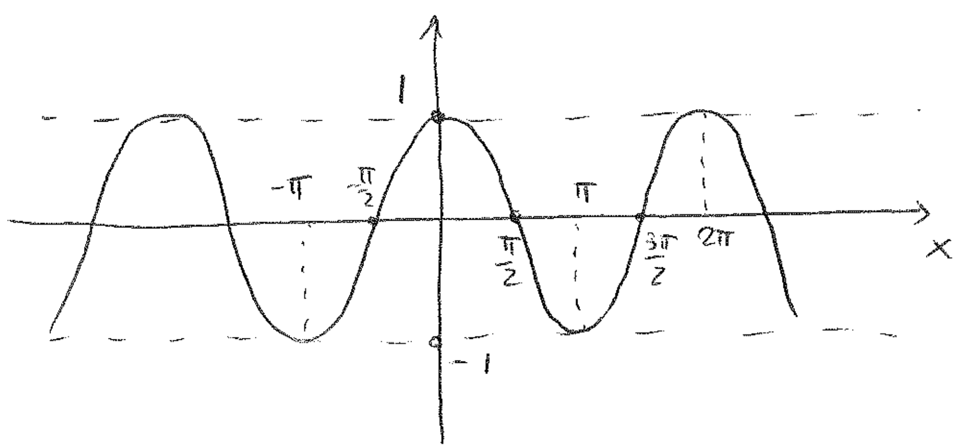
29. What is $\tan(\pi/3)$? $\frac{\sin(\pi/3)}{\cos(\pi/3)} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$

30. What is $\sin(5\pi/6)$? $1/2$

31. What is $\cos(-3\pi/4)$? $-\sqrt{2}/2$



32. Sketch the function $f(x) = \cos(x)$, making sure to mark the x-intercepts and the minima/maxima.



note: that was not supposed to be there but in case you want to know how to do it see

33. 34. Sketch the functions $f(x) = 2\sin(2x) - 1$, making sure to mark ~~the x-intercepts and~~ the minima/maxima.

$\min = -2-1 = -3$

$\max = +2-1 = 1$

period $= \frac{2\pi}{2} = \pi$

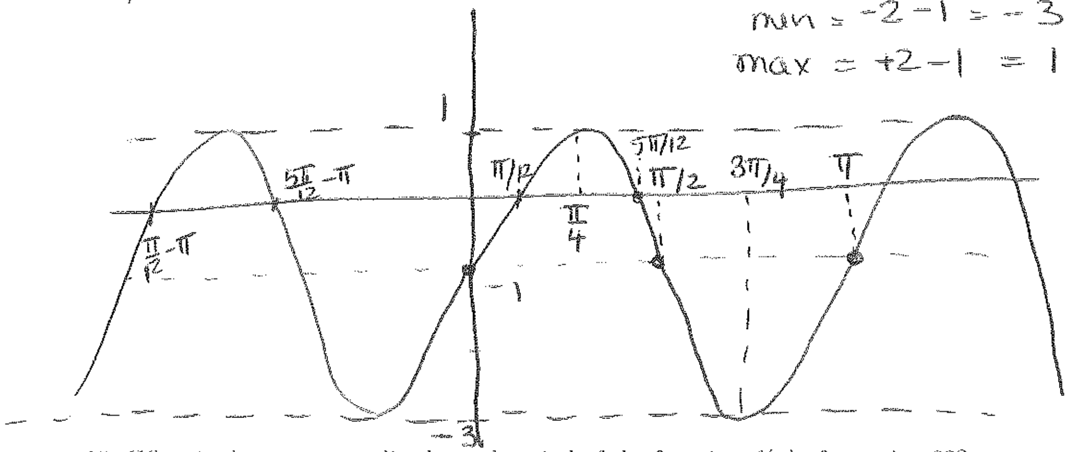
$2\sin(2x) - 1 = 0$

$\sin(2x) = \frac{1}{2}$

$2x = \begin{cases} \frac{\pi}{6} + 2k\pi \\ \frac{5\pi}{6} + 2k\pi \end{cases}$

So

$x = \begin{cases} \frac{\pi}{12} + k\pi \\ \frac{5\pi}{12} + k\pi \end{cases}$



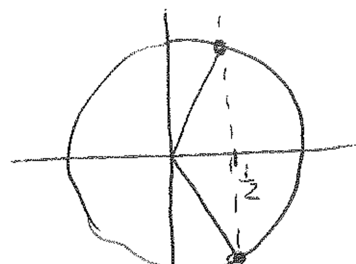
35. What is the mean, amplitude, and period of the function $f(x)$ of question 33?

mean: -1 amplitude: 2 period: π

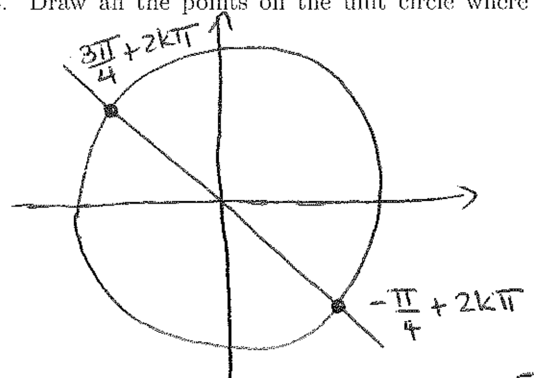
36. Solve the equation $\frac{1}{\cos(x)} = 2$.

$\frac{1}{\cos x} = 2 \rightarrow \cos x = \frac{1}{2}$

$x = \begin{cases} \frac{\pi}{3} + 2\pi k \\ -\frac{\pi}{3} + 2\pi k. \end{cases}$



37., 38. Draw all the points on the unit circle where $\cos(x) = -\sin(x)$, and solve this equation for x.



$x = -\frac{\pi}{4} + 2k\pi$

or $= +\frac{3\pi}{4} + 2k\pi$

note: I had the wrong numbers in

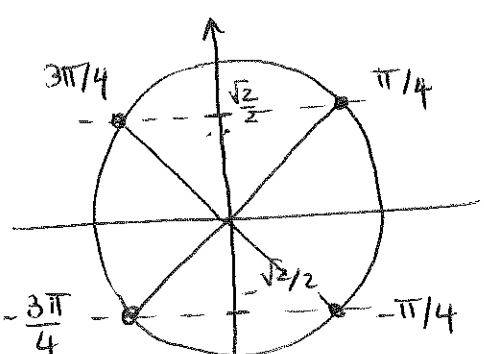
39.,40. Simplify the equation $\cos^2(x) + 3\sin^2(x) = 2$ and solve the equation for x.

$\cos^2 x + \sin^2 x + 2\sin^2 x = 2 \rightarrow 1 + 2\sin^2 x = 2$

$\rightarrow 2\sin^2 x = 2 - 1 = 1$

$\rightarrow \boxed{\sin^2 x = \frac{1}{2}}$

$\rightarrow \sin x = \pm\sqrt{\frac{1}{2}} = \pm\frac{\sqrt{2}}{2}$



$\rightarrow x = \left\{ \frac{\pi}{4} + 2k\pi ; \frac{3\pi}{4} + 2k\pi \right.$

$\left. -\frac{\pi}{4} + 2k\pi ; -\frac{3\pi}{4} + 2k\pi \right\}$

PROBLEM 2: RATIONAL FUNCTIONS. [15 POINTS] Consider the function $f(x) = \frac{1}{x-2} + \frac{1}{(x-3)(x+2)}$.

(a) Reduce to the same denominator, simplify then factor this expression.

$$f(x) = \frac{(x-3)(x+2) + x-2}{(x-3)(x+2)(x-2)} = \frac{x^2+2x-3x-6+x-2}{(x-3)(x+2)(x-2)}$$

$$= \frac{x^2-8}{(x-3)(x+2)(x-2)} = \frac{(x-\sqrt{8})(x+\sqrt{8})}{(x-3)(x+2)(x-2)}$$

(b) What is the domain of $f(x)$? $\mathbb{R} - \{-2, 2, 3\}$

(c) What is the y-intercept? $\frac{1}{-2} + \frac{1}{-6} = \frac{-8}{12} = -\frac{2}{3}$

(d) What are the x-intercepts? $-\sqrt{8}$, $+\sqrt{8}$

(e) Draw a signs table for $f(x)$

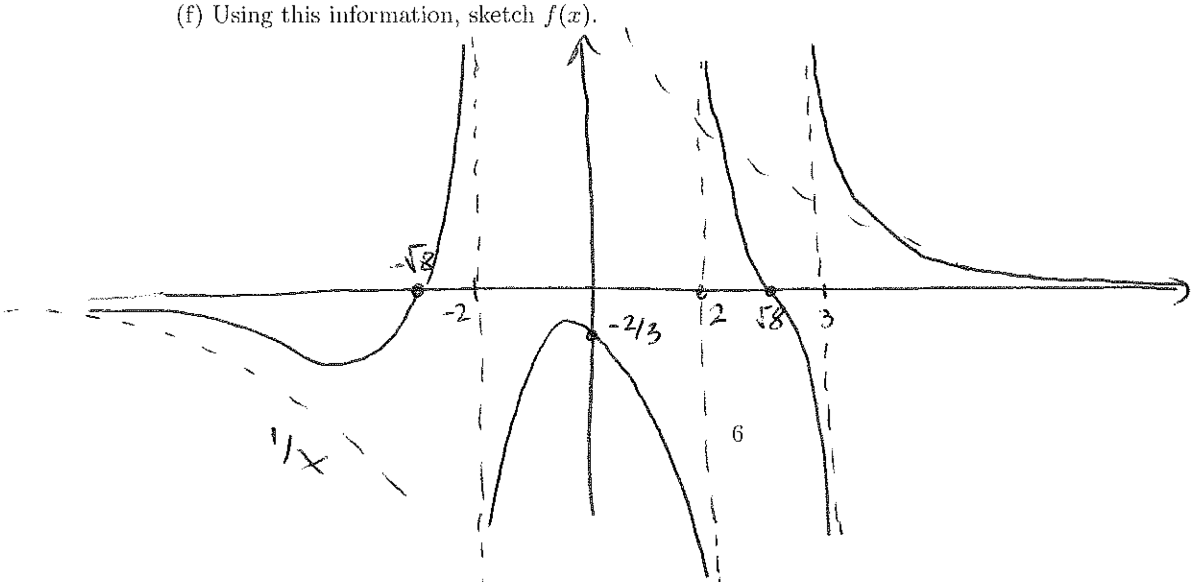
$2 < \sqrt{8} < 3$

		$-\sqrt{8}$		-2		2		$\sqrt{8}$		3	
$x-\sqrt{8}$	$-$		$-$		$-$		$-$	0	$+$		$+$
$x+\sqrt{8}$	$-$	0	$+$		$+$		$+$		$+$		$+$
$x-3$	$-$		$-$		$-$		$-$		$-$	0	$+$
$x-2$	$-$		$-$		$-$	0	$+$		$+$		$+$
$x+2$	$-$		$-$	0	$+$		$+$		$+$		$+$
	$-$	0	$+$	∞	$-$	∞	$+$	0	$-$	∞	$+$

(e) What is the behavior of $f(x)$ as x goes to $+\infty$ and $-\infty$?

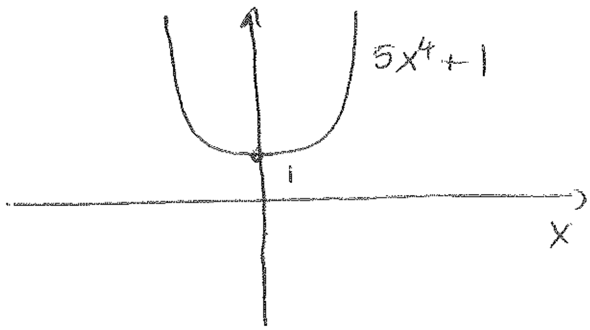
$f(x) \sim \frac{x^2}{x^3} \sim \frac{1}{x}$ so $f(x)$ goes to 0 like $\frac{1}{x}$

(f) Using this information, sketch $f(x)$.



PROBLEM 3: POWER, INVERSE AND COMPOSITION [15 POINTS] Consider the function $f(x) = 5x^4 + 1$.

(a) Sketch this function, making sure to mark the intercepts.



(b) Does this function have an inverse (is this function one-to-one)? YES / NO

(c) If you answered YES, find this inverse of this function. If you answered NO, find an interval where the inverse does exist and calculate the inverse

take $x > 0$ interval then

$$y = 5x^4 + 1 \rightarrow y - 1 = 5x^4 \rightarrow x^4 = \frac{y-1}{5}$$

$$\rightarrow x = \left(\frac{y-1}{5}\right)^{1/4}$$

So $f^{-1}(x) = \left(\frac{x-1}{5}\right)^{1/4}$

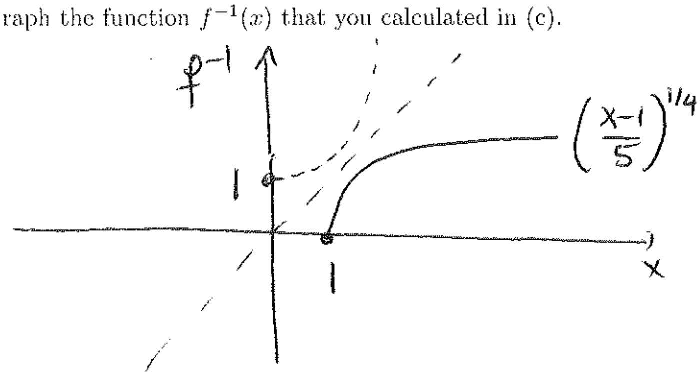
(d) Using your solution from question (c), verify that $f[f^{-1}(x)] = x$

$$5\left(\left(\frac{x-1}{5}\right)^{1/4}\right)^4 + 1 = 5\left(\frac{x-1}{5}\right) + 1$$

$$= x - 1 + 1$$

$$= x$$

(e) Graph the function $f^{-1}(x)$ that you calculated in (c).



(f) What is the solution of the equation $f(x) = 11$?

$$x = f^{-1}(11) = \left(\frac{11-1}{5}\right)^{1/4} = \left(\frac{10}{5}\right)^{1/4} = 2^{1/4}$$

Applied Problem 1. Logarithms and exponentials. [15 points]

We consider here the function

$$M(n) = 1000 \times (1.1)^n$$

that represents the amount of money in a bank account (in dollars) as a function of the number of years n after investment.

(1) What the value of the initial investment? 1000

What is the yearly interest rate? 0.1 (10%)

(2) What is amount of money in the account after 1 year? $1000 \times 1.1 = 1100$

What is amount of money in the account after 2 years? $1100 \times 1.1 = 1100 + 110 = 1210$

(3) Write $M(n)$ in the form of an exponential in base e: $1000\, e^{n \ln(1.1)}$

(4) What is the doubling time of the function $M(n)$? (Note: you can either answer this question by remembering the formula for the doubling time, or by solving a particular equation).

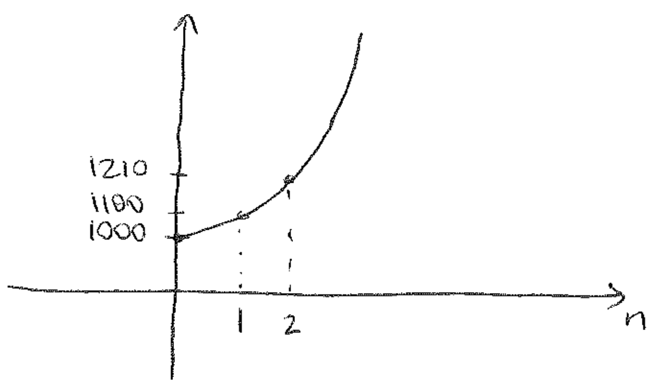
Solution 1: doubling time is $\frac{\ln 2}{r} = \frac{\ln 2}{\ln 1.1}$

Solution 2: $2000 = 1000\,(1.1)^n \to 2 = (1.1)^n$

$\to \ln 2 = n \ln(1.1)$

$\to n = \ln 2 / \ln(1.1)$

(5) Sketch the function $M(n)$ (no need to be too precise, but do include the answers to questions 1 and 2)

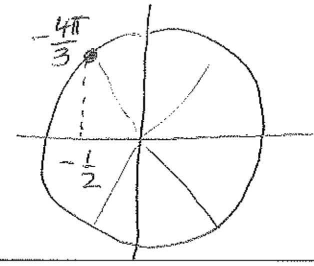


(6) How many years would one have to wait to get $10,000 in the account?

$10000 = 1000\,(1.1)^n \to \frac{10000}{1000} = (1.1)^n = 10$

$\to \ln[(1.1)^n] = \ln 10 \to n \ln(1.1) = \ln 10$

$\to n = \frac{\ln 10}{\ln(1.1)}$



$-\frac{8\pi}{6} = -\frac{4\pi}{3}$

$\cos\left(-\frac{8\pi}{6}\right) = -\frac{1}{2}$

APPLIED PROBLEM 2. PERIODIC FUNCTIONS [15 POINTS]

Tides are a regular oscillation of the Ocean's mean water height, a phenomenon rather well-known by Santa Cruz surfers. The water height (ignoring waves) varies with time as

$$H(t) = 10 + 1.5\cos\left(\frac{\pi(t-8)}{6}\right)$$

where t is given in hours (t=0 hours being midnight), and H is given in meters.

(a) What is the water height at midnight? $H(0) = 10 + 1.5\cos\left(-\frac{8\pi}{6}\right) = 10 - \frac{1.5}{2} = 9.25$

(b) What is the difference in water height between high-tide (i.e. when the height is maximum) and low-tide (i.e. when the height is minimum)?

twice the amplitude, i.e, $\boxed{3}$ meters.

(c) What is the period of the tide? $p = \frac{2\pi}{\frac{\pi}{6}} = 12$ hours

(d) At what times of the day is high-tide? High tide is $10 + 1.5 = 11.5 \rightarrow$

solve $H(t) = 11.5 \rightarrow 10 + 1.5\cos\left(\frac{\pi(t-8)}{6}\right) = 11.5$

$\rightarrow \cancel{1.5}\cos\left(\frac{\pi(t-8)}{6}\right) = \cancel{1.5} \rightarrow \cos\left(\frac{\pi(t-8)}{6}\right) = 1$

$\rightarrow \frac{\pi(t-8)}{6} = 0, 2\pi, \text{etc} \rightarrow t = 8, \quad t = 12 + 8 = 20$ etc.

(e) Based on this, sketch the function $H(t)$ for t between 0 and 24.