Algebra Workshop 3: Rational expressions

1 Simplfying rational expressions by factoring

RULE:

- Rational expressions in expanded form cannot be simplified directly. To simplify a rational expression, it must first be factored. Then, if factors cancel out, then the expression can be simplified.
- Remember that
$$\frac{a-b}{b-a} = -1$$
- Remember that for any expression E (which could be a function of x)
$$\frac{E^n}{E^m} = E^{n-m}$$

EXAMPLE 1

$$\frac{x^2-4}{(x-2)^2} = \frac{(x-2)(x+2)}{(x-2)^2} = \frac{x+2}{x-2}$$

provided $x \neq 2$. Remember that you cannot simplify x^2 and x, or 4 and 2 in the original expression. That would not make mathematical sense.

EXAMPLE 2

$$\frac{3-x}{x^2-x-6} = \frac{3-x}{(x-3)(x+2)} = -\frac{1}{x+2}$$

PRACTICE: Simplify the following expressions as much as possible.

- $\frac{2-x}{x(x-2)} = \frac{-\cancel{(x-2)}}{x\cancel{(x-2)}} = -\frac{1}{x}$
- $\frac{2x+2}{2x-6} = \frac{2(x+1)}{2(x-3)} = \frac{x+1}{x-3}$
- $\frac{5x+30}{x^2+6x} \longrightarrow = \frac{5\cancel{(x+6)}}{x\cancel{(x+6)}} = \frac{5}{x}$
- $\frac{x^2-x-12}{16-x^2} = \frac{(x+3)(x-4)}{(4-x)(4+x)} = \longrightarrow \frac{(x+3)\cancel{(x-4)}}{-\cancel{(x-4)}(4+x)} = -\frac{x+3}{4+x}$
- $\frac{5x-2}{4-25x^2} = \frac{5x-2}{(2-5x)(2+5x)} = \frac{-\cancel{(2-5x)}}{\cancel{(2-5x)}(2+5x)} = -\frac{1}{2+5x}$
- $\frac{x^3+4x^2-3x-12}{(x+4)^2}$ $\rightarrow \frac{x^2(x+4)-3(x+4)}{(x+4)^2} = \frac{\cancel{(x+4)}(x^2-3)}{(x+4)\cancel{(x+4)}} = \frac{x^2-3}{x+4}$

2 Multiplying rational expressions the efficient way

When multiplying rational expressions, it is tempting to just multiply out the respective numerators and denominators. However, in some cases, it pays off to first factor expressions, and see if there are cancellations before expanding things out...

EXAMPLE 1:

$$\frac{x^2}{x+3} \cdot \frac{x+5}{x-7} = \frac{x^2(x+5)}{(x+3)(x-7)} = \frac{x^3+5x^2}{x^2-4x-21}$$

if we need the expression in expanded form. In this example, there was no obvious cancellation in the factored form, so we can indeed just multiply out the factors. But compare with

EXAMPLE 2:

$$\frac{x+3}{x-4}\cdot\frac{x^2-2x-8}{x^2-9}=\frac{(x+3)(x^2-2x-8)}{(x-4)(x^2-9)}=\frac{(x+3)(x-4)(x+2)}{(x-4)(x+3)(x-3)}=\frac{x+2}{x-3}$$

If instead we had multiplied out the factors in step 2, we would never have found this simple expression...

PRACTICE:

- $\frac{x-3}{x+7}\cdot\frac{3x+21}{2x-6}$ $= \frac{\cancel{(x-3)}\,3\cancel{(x+7)}}{\cancel{(x+7)}\,2\cancel{(x-3)}} = \frac{3}{2}$
- $\frac{x^2-49}{x^2-4x-21}\cdot\frac{x+3}{x}$ $= \frac{\cancel{(x-7)}(x+7)}{\cancel{(x-7)}\cancel{(x+3)}}\cdot\frac{\cancel{x+3}}{x} = \frac{x+7}{x}$
- $\frac{6y+2}{y^2-1}\cdot\frac{1-y}{3y^2+y}$ $= \frac{2\cancel{(3y+1)}}{(y-1)(y+1)}\cdot\frac{1-y}{\cancel{(3y+1)}y} = \frac{-2\cancel{(y-1)}}{\cancel{(y-1)}(y+1)y} = -\frac{2}{y(y+1)}$
- TYPO → $(x+1)\cdot\frac{x+2}{x^2+5x+6}$ $= (x+1)\frac{\cancel{x+2}}{(x+3)\cancel{(x+2)}} = \frac{x+1}{x+3}$
- $\frac{5x+5}{7x-7x^2}\cdot\frac{2x^2+x-3}{4x^2-9}$ $= \frac{5(x+1)}{7x(1-x)}\cdot\frac{\cancel{(2x+3)}(x-1)}{(2x-3)\cancel{(2x+3)}} = \frac{+5(x+1)\cancel{(x-1)}}{-7x\cancel{(x-1)}(2x-3)} = -\frac{5(x+1)}{7x(2x-3)}$

3 Divisions of rational expressions by one another

Remember the one rule to simplify compound rational expressions (compound fractions): Flip them up and Multiply. Then, just use the method learned in the previous section.

EXAMPLE 1:

$$\frac{4x^2-25}{\frac{2x+5}{14}}=\frac{4x^2-25}{1}\cdot\frac{14}{2x+5}=\frac{14(4x^2-25)}{2x+5}=\frac{14(2x+5)(2x-5)}{2x+5}=14(2x-5)$$

EXAMPLE 2:

$$\frac{\frac{x^2+3x-10}{2x}}{\frac{x^2-5x+6}{x^2-3x}}=\frac{x^2+3x-10}{2x}\cdot\frac{x^2-3x}{x^2-5x+6}=\frac{(x^2+3x-10)(x^2-3x)}{2x(x^2-5x+6)}=\frac{(x+5)(x-2)x(x-3)}{2x(x-3)(x-2)}=\frac{x+5}{2}$$

PRACTICE:

- $\frac{\frac{x+5}{7}}{\frac{4x+20}{9}}$ $= \frac{x+5}{7}\cdot\frac{9}{4x+20} = \frac{\cancel{(x+5)}}{7}\,\frac{9}{4\cancel{(x+5)}} = \frac{9}{28}$
- $\frac{\frac{7}{y-5}}{\frac{28}{3y-15}}$ $= \frac{7}{y-5}\cdot\frac{3y-15}{28} = \frac{\cancel{7}}{\cancel{y-5}}\cdot\frac{3\cancel{(y-5)}}{4\times\cancel{7}} = \frac{3}{4}$
- $\frac{\frac{y^2+y}{y^2-4}}{\frac{y^2-1}{y^2+5y+6}}$ $= \frac{y^2+y}{y^2-4}\cdot\frac{y^2+5y+6}{y^2-1} = \frac{y\cancel{(y+1)}}{(y-2)\cancel{(y+2)}}\,\frac{\cancel{(y+2)}(y+3)}{(y-1)\cancel{(y+1)}}$ $= \frac{y(y+3)}{(y-2)(y-1)}$
- $\frac{x^2+4x-5}{\frac{x^2-25}{x+7}}$ $= (x^2+4x-5)\frac{x+7}{(x-5)(x+5)} = \cancel{(x+5)}(x-1)\frac{x+7}{(x-5)\cancel{(x+5)}}$ $= \frac{(x-1)(x+7)}{x-5}$

Algebra Workshop 4: Expressions with exponents

1 Rules for expressions with exponents

Review the following rules:

- $a^m a^n = a^{m+n}$ and $a^m a^{-n} = a^{m-n}$
- $(a^m)^n = a^{mn} = (a^n)^m$
- $a^m/a^n = a^{n-n}$
- $(ab)^m = a^m b^m$
- $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$
- $a^{-m} = \frac{1}{a^m}$
- $a^0 = 1$
- $\left(\frac{a}{b}\right)^{-m} = \left(\frac{b}{a}\right)^m$

NOTE: These rules are applicable whether one has integer exponents, rational or irrational exponents.

PRACTICE: Each group gives one example of application for each of these 8 rules. Try to mix and match which "direction" you illustrate the rule. Be creative!

IMPORTANT OTHER RULES:

- $\sqrt{-5} = (-5)^{1/2}$ is not a real number - and the same applies for any even root of a negative number! But on the other hand $\sqrt[3]{-5} = (-5)^{1/3}$ exists and is equal to $-\sqrt[3]{5}$. This is true for any odd root of a negative number.
- $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$
- $\sqrt{x^2} = |x|$ and not $\sqrt{x^2} = x$. The same is true for all even root.
- $\sqrt[3]{x^3} = x$. The same is true for all odd roots.

PRACTICE: TRUE OR FALSE?

- $\sqrt{49} = -7$ F
- $\sqrt[3]{-27} = -3$ T
- $\left(\sqrt[3]{10}\right)^3 = 10$ T
- $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$ F
- $\sqrt{xy} = \sqrt{x}\sqrt{y}$ T
- $\sqrt{x^2 - y^2} = x - y$ F

- $\frac{\sqrt{x}}{\sqrt{y}} = \sqrt{\frac{x}{y}}$ T
- $\sqrt[3]{p^3+q^3} = \sqrt[3]{p^3} + \sqrt[3]{q^3}$ F
- $\sqrt{10+6} = \sqrt{10} + \sqrt{6}$ F
- $\sqrt{10+6} = 4$ T
- $\sqrt{a^2} - a = 0$ T if $a > 0$
- $\sqrt[5]{b^5} - b = 0$ T

2 Simplifying simple expressions

- $(1+x^2)^2(x^2+1)^7 =$ $(x^2+1)^9$
- $\frac{3^x}{3^{1-x}} =$ $3^{x-(1-x)} = 3^{2x-1}$
- $\left(2x^2y^3z\right)^2 =$ $4x^4y^6z^2$
- $\left(\frac{a}{b^{1/2}}\right)^{-1} =$ $\frac{b^{1/2}}{a}$
- $\left(\frac{x^3y^2z}{xy^2z^{-3}}\right)^{-3} =$ $\left(\frac{x\cancel{y^2}z^{-3}}{x^3\cancel{y^2}z}\right)^3 = \left(\frac{x^{1-3}}{z^{1+3}}\right)^3 = \frac{(x^{-2})^3}{(z^4)^3} = \frac{1}{x^6z^{12}}$
- $2^n8^{n+1} =$ $2^n\left(2^3\right)^{n+1} = 2^n\,2^{3(n+1)} = 2^{4n+3}$
- $\left(2x^{1/2}y^{1/4}z\right)^2 =$ $4\,x\,y^{1/2}z^2$
- $\left(\frac{a}{a^{1/2}}\right)^2 =$ $\frac{a^2}{a} = a$
- $\left(\frac{x^{3/2}y^{1/2}z}{xy^{3/2}z^{-3/2}}\right)^4 =$ $\left(x^{3/2-1}\,y^{1/2-3/2}\,z^{1+3/2}\right)^4 = \left(x^{\frac{1}{2}}\,y^{-1}\,z^{5/2}\right)^4 = x^2\,y^{-4}\,z^{10}$
- $4^{1/n}8^{\frac{1}{2n}} =$ $\left(2^2\right)^{1/n}\left(2^3\right)^{\frac{1}{2n}} = 2^{\frac{2}{n}}\,2^{\frac{3}{2n}} = 2^{\frac{2}{n}+\frac{3}{2n}} = 2^{\frac{4+3}{2n}} = 2^{\frac{7}{2n}}$

3 Simplifying harder expressions

- Simplify $(-1-2x-x^2)^3(x^2+2x+1)^{-3} =$ $\frac{\left((-1)(1+2x+x^2)\right)^3}{(1+2x+x^2)^3} = (-1)^3\,\frac{\cancel{(1+2x+x^2)^3}}{\cancel{(1+2x+x^2)^3}} = -1$
- Simplify $\frac{\sqrt{1+x^2}}{(1+x^2)}$ $= \frac{1}{\sqrt{1+x^2}}$
- Simplify $\frac{(1-x^2)^{5/2}}{(1-x^2)^{3/2}}$ $= (1-x^2)^{5/2-3/2} = (1-x^2)$
- Simplify $\frac{2^{2x}-1}{2^x+1}$ $= \frac{(2^x)^2-1}{2^x+1} = \frac{(2^x-1)\cancel{(2^x+1)}}{\cancel{2^x+1}} = 2^x-1$
- Simplify $(3^{x/2}+3^{-x/2})(3^{x/2}-3^{-x/2})$

↳ looks like $(a+b)(a-b) \rightarrow = a^2-b^2$

$= \left(3^{x/2}\right)^2 - \left(3^{-x/2}\right)^2 = 3^x - 3^{-x}$