Algebra Workshop 2: Factoring polynomials

1 The standard expressions

THE THREE FORMULAE YOU HAVE TO KNOW: for any expression "E" and "F",

- $E^2 + 2EF + F^2 = (E + F)^2$
- $E^2 - 2EF + F^2 = (E - F)^2$
- $E^2 - F^2 = (E - F)(E + F)$

EXAMPLES:

- $x^2 + 6x + 9 = (x + 3)^2$
- $-x^2 + 2x - 1 = -(x^2 - 2x + 1) = -(x - 1)^2$
- $10 - x^2 = (\sqrt{10} - x)(\sqrt{10} + x)$

NOTE: This is not limited to these simple examples...

- $-x^4 - 4x^2 - 4 = -(x^4 + 4x^2 + 4) = -(x^2 + 2)^2$
- $x^6 - 5x^3 + \frac{25}{4} = \left(x^3 - \frac{5}{2}\right)^2$
- $(10 - x)^2 - (2 + 2x)^2 = [(10 - x) + (2 + 2x)][(10 - x) - (2 + 2x)] = [12 + x][8 - 3x]$

PRACTICE: Recognize one of the standard formulae, and factor the polynomial using this formula. If the polynomial is not one of the three standard formula above, mark it with a star. Hint: in total, there are 4 starred problems.

- $x^2 + 12x + 36$
- $x^2 - 2x + 1$
- $x^2 + 4x - 4$
- $-x^2 + 3x - \frac{9}{4}$
- $2x^2 + 8x + 6$
- $5x^2 - 20x + 20$
- $x^2 - 11$
- $1 - x^4$
- $x^3 + 1$
- $-2x^2 + 5$
- $(3x + 1)^2 + x^2$
- $(2x - 1)^2 - 2x^2$
- $x^2 + 2x + 1 - 2x^2 - 8x - 8$

PRACTICE: Invent two "hard" problems which can be factored using these standard expressions, and give them to your working partner. Solve the problems, and check with your TA/teacher.

2 Common factor

Rule: In a sum of terms, if you recognize a "common factor", you can factor it out of every term.

Examples:

- $2x^3 - 2x = 2x(x^2 - 1)$ (the factor $2x$ is in both $2x^3$ and $2x$)... Then $x^2 - 1$ can be factored further...
- $-30x^3 + 27x^2 - 9 = -3(10x^3 - 9x^2 + 3)$... Here, 3 is the only common factor, and it "looks nicer" by factoring a minus too.
- $6x^4 + 36x^2 = 6x^2(x^2 + 6)$

Note: This is not limited to these simple examples...

- $(x-2)(x+1)^2 - (x+1)(x+3) = (x+1)[(x-2)(x+1) - (x+3)] = (x+1)[x^2 - x - 2 - x - 3] = (x+1)(x^2 - 2x - 5)$
- $x^2 - 16 + (x+1)(4-x) = (x-4)(x+4) + (x+1)(4-x) = (x-4)(x+4) - (x+1)(x-4) = (x-4)[(x+4) - (x+1)] = 3(x-4)$

Practice: Find a common factor and ... factor.

- $-2x^3 - 2x^2 + 4x$
- $2x^{10} + 3x^5 + 4x^7 - 2x^2$
- $-x(x+1) + (x-2)(x+1) - 2(1+x)$
- $121x^2 + 11x - 77$
- $(x-2)^2 + (x-2)(10+x)$
- $(4+x)(2-x) + 2x - 4$
- $x^4 - 1 + 6(x^2 + 1)$

3 Grouping

The grouping technique only works occasionally, for expressions which have 4 terms or more (no point "grouping" less than 4 terms). The idea is that finding a common factor between two pairs of terms is easier than finding a common factor in the whole expression. Then, sometimes, each *pair* once factored contains a term which is common to all pairs, and can be used as the next common factor.

Examples:

- $x^3 + 2x^2 - x - 2 = (x^3 + 2x^2) - (x+2) = x^2(x+2) - (x+2) = (x+2)(x^2 - 1) = (x+2)(x-1)(x+1)$
- $2x^5 - 3x^4 + 6x^2 - 9x = (2x^5 - 3x^4) + (6x^2 - 9x) = x^4(2x-3) + 3x(2x-3) = (x^4 + 3x)(2x-3) = x(x^3+3)(2x-3)$

Note: Usually, there are different possible ways of grouping terms which have common factors. Starting with a different pair still yields the same result.

Practice: For the two examples above, find a different way of pairing the terms and redo the problem to verify you obtain the same answer.

More grouping problems: Group these problems in as many different ways as possible

- $2x^8 - 4x^2 - 6x^6 + 12$
- $x^{11} + x^9 - 2x^7 - 2x^5 + 6x^3 + 6x$