Review Worksheet

This worksheet will help you prepare for the Final.

Studying tips: Read each section. If you think you master the material, try to do the sample problems from the Practice Final to convince yourself you know how to do them. If you don't think you master the material, first go through the textbook (the study plan is a great place to start) and lecture notes about the material, make sure you understand the examples. Then do as many sample problems as you can in the Study Plan, and finally, try the Practice Final. The bottom line is: don't spend too much time on the points you know - make sure you spend most of your time studying for the points you're less familiar with.

If you have any difficulties, write them down, come to office hours to get help.

The final is comprehensive and will include all class and section material.

1 Basic algebra

Numbers: You need to know the values of a few important numbers:

- $\pi \simeq 3.1$

- $e \simeq 2.7$

- $\sqrt{2} \simeq 1.4$

- $\sqrt{3} \simeq 1.7$

Intervals

- You need to know the correct notations for intervals, in particular the difference between open and closed intervals. Also be sure to know how to write intervals which extend to either $+\infty$ or $-\infty$.

- You need to be able to write intervals as inequalities, and vice-versa

Exponents: You need to be completely comfortable manipulating expressions with exponents. This involves *knowing* the following formulae (i.e. knowing where they come from, and how to use them without mistake). For any expression E, and any number a or b:

$$E^0 = 1$$
$$E^{a+b} = E^a E^b$$
$$E^{-a} = \frac{1}{E^a}$$
$$E^{a-b} = \frac{E^a}{E^b}$$
$$E^{ab} = (E^a)^b = (E^b)^a$$

Note: don't fall into the standard traps:

- $(E^a)^b$ IS NOT E^{a+b}.

- $(E + F)^a$ IS NOT $E^a + F^a$

Factoring: You need to know how to factor an expression with respect to a particular variable. For this purpose, you should think about this approach to the problem:

- Is the expression one of the standard formulae for factoring?

   1. Is it a difference of squares $a^2 - b^2$? In that case factor as $(a - b)(a + b)$
   2. Is is something like $a^2 + 2ab + b^2$? In that case factor as $(a + b)^2$
   3. Is is something like $a^2 - 2ab + b^2$? In that case factor as $(a - b)^2$

- Is it a quadratic expression of the kind $ax^2 + bx + c$? If that's the case, see the Section on Quadratics.

- If it's not a standard expression, or a quadratic, is there an obvious common factor? If so, begin by factoring it out, and then deal with the next expressions using the same chart.

- If you can't see a common factor, can you group terms in pairs (or sometimes triplets) which can each be factored ? If that's the case, try that, and see if the remaining factors then become a "common factor"

2   Definition and basic use of a function

- You need to know how to identify that a rule is, or isn't a function

- You need to know how to determine that a graph is, or isn't the graph of a function (i.e. vertical line test)

- You need to know that every point on the graph of a function has coordinates $(x, f(x))$

- You need to be able to determine the domain of definition of a given function

- You need to be able to evaluate functions at any given point (e.g. $f(2), f(x^2), f(x + h), f(a)$)

- You need to know the basic operations on functions (sum, product, quotient, ...)

- You need to know how to compose functions $f \circ g(x) = f[g(x)]$ and $g \circ f = g[f(x)]$

- You need to know how to recognize odd and even functions.

3   Graphs of basic functions

Graphs of basic functions: You need to know how to *sketch* the following functions without using a calculator. Sketching means "to draw something quickly, without too much attention to detail or perfect accuracy, but nevertheless represent any salient feature correctly".

- $f(x) = $ a constant (e.g. $f(x) = 2$, $f(x) = -\pi$)

- $f(x) = ax + b$

- $f(x) = |x|$

- $f(x) = x^2$, $f(x) = x^3$, $f(x) = x^4$, $f(x) = x^5$, etc...

- $f(x) = \sqrt{x}$

- $f(x) = \frac{1}{x}$, $f(x) = \frac{1}{x^2}$, $f(x) = \frac{1}{x^3}$, $f(x) = \frac{1}{x^4}$, etc...

- $f(x) = \ln(x)$, $f(x) = e^x$, $f(x) = \log_a(x)$, $f(x) = a^x$

- $f(x) = \sin(x)$, $f(x) = \cos(x)$ and $f(x) = \tan(x)$.

Graphs of functions which are translated vertically and horizontally: Given the graph of a function $f(x)$, you have to know how to graph the functions

- $f(x) + a$: the graph is moved up or down (if a is positive or negative respectively) by an amount a

- $f(x + a)$: the graph is moved left or right (if a is positive or negative respectively) by an amount a

Graphs of functions which are reflected across the $x-$axis and $y-$axis: Given the graph of a function $f(x)$, you have to know how to graph the functions

- $-f(x)$: the graph is reflected across the $x-$axis

- $f(-x)$: the graph is reflected across the $y-$axis

4 Quadratic functions

For a given function $f(x) = ax^2 + bx + c$ you need to know

- how to find the $y-$intercept (e.g. the $y-$intercept is c)

- how to complete the square to transform $f(x)$ into the vertex form $f(x) = a(x - x_V)^2 + y_V$.
$f(x) = a\left(x^2 + \frac{b}{a}x\right) + c = a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2}\right) + c = a\left(x + \frac{b}{2a}\right)^2 - a\frac{b^2}{4x^2} + c$

- You need to know and understand from basic transformations of the $y = x^2$ parabola *why* $\left(x_V = -\frac{b}{2a}, y_V = f(x_V)\right)$ are the coordinates of the vertex of the parabola.

- based on the sign of a, whether the parabola opens up $(a > 0)$ or down $(a < 0)$

- how to find $x-$intercepts (e.g. how to solve the equation $ax^2 + bx + c = 0$) using the discriminant method, see Handout 1.

- how to factor the quadratic depending on the discriminant (see Handout 1)

- how to draw a signs table for the quadratic, and how the signs table differs in the case $D = b^2 - 4ac$ is positive, zero and negative.

- how the signs table relates to the graph of the function f.

5 Higher-order polynomials

You need to know

- How to expand a polynomial, and recognize the leading-order term (the $a_n x^n$ term).

- That the behavior of the graph of the polynomial for very large x is the same as the behavior of the leading order term. How to deduce from that whether $f(x)$ goes to $+\infty$ or $-\infty$ as x goes to $+\infty$ or $-\infty$.

- How to recognize that a polynomial is fully factored or not

- How to factor it, if it is not fully factored.

- How to draw a signs table for the polynomial

- How to deduce from the signs table what the shape of the graph is.

- How to find the behavior near a root.

6   Rational functions

You need to know

- What the numerator and denominator are.

- How to reduce an expression to a rational function if necessary

- How to find the domain of definition of a rational function (i.e. exclude points where the denominator is 0).

- How to determine the behavior of the graph of the rational function for very large x (i.e. to look at the leading order terms in the numerator and in the denominator).

- What are, and how to find vertical, horizontal and oblique asymptotes.

- How to recognize that a rational function is fully factored or not

- How to factor it, if it is not fully factored, and how to simplify it.

- What to do with "excluded points", points where both denominator and numerator have a root (i.e. to draw a circle where the point should be)

- How to draw a signs table for the rational function (remember the asymptotes).

- How to deduce from the signs table what the shape of the graph is. In particular, how to deduce what the shape of the graph near the asymptotes are.

7   Inequalities

You need to know:

- how to manipulate very simple inequalities, and the basic rules, including:
  - adding and substracting a number on both sides does not change the direction of the inequality
  - multiplying/dividing by a positive number on both sides does not change the direction of the inequality
  - multiplying/dividing by a negative number on both sides *changes* the direction of the inequality

- when applying the same rule on both sides of an inequality, whether the direction of the inequality changes or not depends on the slope of the graph of the rule. For example:
  - applying a rule with positive slope (increasing function) does not change the direction of the inequality
  - applying a rule with negative slope (decreasing function) changes the direction of the inequality
  - if the rule sometimes increases, sometimes decrases, you cannot know.

- how to solve an inequality using basic algebraic manipulations and signs tables.

8 Power laws

- You need to know the basic grahs of all power laws ($f(x) = x^a$) for every possible value of a. This includes

  - Knowing the domain of definition
  - Knowing the behavior near infinity (when $x \to -\infty$ and $x \to +\infty$)
  - Knowing whether there are asymptotes, and what is the behavior near the asymptote
  - Knowing how to construct functions from that $x^a + b$, bx^a, $(x - b)^a$

- You need to know the formulas for the manipulations of power laws

- You need to know how to find the inverses of the power functions, and solve basic equations involving these powers.

9 Inverse of functions

- You need to know how to determine graphically that a function has an inverse (i.e. horizontal line test)

- You need to know the notation: the inverse of $f(x)$ is the function denoted by $f^{-1}(x)$.

- You need to know that $f^{-1}(x)$ DOES NOT mean $\frac{1}{f(x)}$.

- You need to be able to calculate the inverse of simple functions (i.e. solve the equation $y = f(x)$ for x.

- You need to know that the inverse f^{-1} applied to f yields x, and same for f applied to f^{-1}: $f[f^{-1}(x)] = f^{-1}[f(x)] = x$. You need to know how to use this fact to verify that the inverse you calculated is correct.

- You need to know and understand the relationship between the graph of a function and the graph its inverse (i.e. they are mirror images with respect to the $y = x$ line). You need to be able to use that knowledge to find the graph of f^{-1} based on the graph of $f(x)$.

10 Exponentials and logarithms

Exponentials: You need to know

- The basic expression for an exponential function $f(x) = a^x$ and NOT mix it up with a power function.

- The graphs of basic exponentials including the natural exponential, both in the form $f(x) = a^x$, $f(x) = a^{-x}$ and $f(x) = \left(\frac{1}{a}\right)^x$. This includes the horizontal asymptote (at $y = 0$) and the $y-$intercept $f(0) = 1$.

- How these graphs change when manipulating the exponential a^{x+b}, $b \times a^x$, $a^x + b$, etc...

- How to manipulate these exponentials (see Formulas sheet) and simplify expressions containing exponentials.

- What the natural exponential is and how to change base from any exponential to the natural exponential

Logarithm: You need to know

- That the logarithm in base a is the inverse of the exponential in base a

- The consequences of that fact in terms of

  - Special values of the log: $\log_a(1) = 0$ and $\log_a(a) = 1$ for any a
  - The graphs of all logarithmic functions: you should be able to graph any basic log function, including the special points above, and the vertical asymptote at $x = 0$.
  - The graphs of related functions $\log_a(x + b)$, $b\log_a(x)$, etc.
  - The domain of the log functions (logs are not defined for $x \leq 0$ so the domain is $x > 0$)
  - The two fundamental equations: $a^{\log_a(x)} = x$ and $\log_a(a^x) = x$.

- How to manipulate logs, and in particular the log properties (see Formula Sheet) and how to use them.

- How to change base between the natural log and other logs.

- How to evaluate the logs of simple numbers.

- What the natural logarithm is and how it relates to the natural exponential.

and generally: you need to know how to solve equations involving logarithms and exponentials.

11 Trigonometric functions

Basic angles, basic trigonometric functions, and the unit circle:

- You need to know the basic angles, and in particular:

  - how to go back and forth between radian and degrees for an angle
  - what the unit circle is, and how to visualize angles on the unit circle
  - that angles are defined within $2k\pi$.

- You need to know the two basic functions $\sin(a)$, $\cos(a)$ and derived functions such as $\tan(a)$, $\sec(a)$, $\cot(a)$ and $\csc(a)$ and more specifically:

  - how to visualize $\cos(a)$ and $\sin(a)$ on the unit circle
  - *why* it works
  - that it implies $\cos^2(a) + \sin^2(a) = 1$, *why*, and how to use this formula.
  - the values of sin and cos for the angles $0, \pi/6, \pi/4, \pi/3, \pi/2$ and all other angles which can by constructed from these ones by symmetry.
  - How to deduce $\tan(a)$, $\sec(a)$, $\cot(a)$ and $\csc(a)$ from these values

- You need to be able to simplify simple trigonometric expressions which use these functions.

- You need to be able to graphs and annotate the graphs of

  - $\cos(x)$
  - $\sin(x)$
  - $\tan(x)$

Periodic functions and inverse trigonometric functions.

- You need to be able to deduce from these graphs the graphs of the inverse functions $\cos^{-1}(x)$, $\sin^{-1}(x)$ and $\tan^{-1}(x)$

- You need to know the definition of a periodic function, and how to find the period of $\cos(bx + c)$ and $\sin(bx + c)$

- You need to be able to sketch the graphs of functions derived from cos and sin such as $A\cos(bx + c)$ and $A\sin(bx + c)$, and solve basic equations that use the inverses.

Trigonometric formulas and trigonometric equations. You need to know

- The Pythagorean formula

- The double angle formulas for sin and cos

- How to use the addition formulas to expand $\cos(a + b)$ and similar expressions, and to find products of expressions such as $\cos(a)\cos(b)$, etc..

- How to solve trigonometric equations (study the various examples given in class/homework).

12 Applied problems

The Final will include one applied problem, similar to the ones given in the Homeworks and/or the problems discussed in class and in section. Remember that to solve applied problems:

- Identify what variables describe the problem. Give them names..

- Identify *all* the clues which relate the variables to one another. Write them as equations.

- Usually, you can then solve for one of the variables in one clue, then use that solution in another clue, or in what you are looking for, in order to make progress.

Solving word-problem is detective work! You need to use everything you know, and be creative!