Practice Final 2, MATH 3

Name: ________________________________

Calculators are not allowed.

Read all the questions before you start working on any of them. Start with the ones you are most comfortable with, and continue with the other ones later. Always double-check your answers.

Relax, and do your best!

PROBLEM 1: SHORT QUESTIONS. [40 POINTS]

1. What is the equation of the line parallel to $y = -2x$ which goes through the point $\overset{A}{(1, 2)}$?

Slope: -2

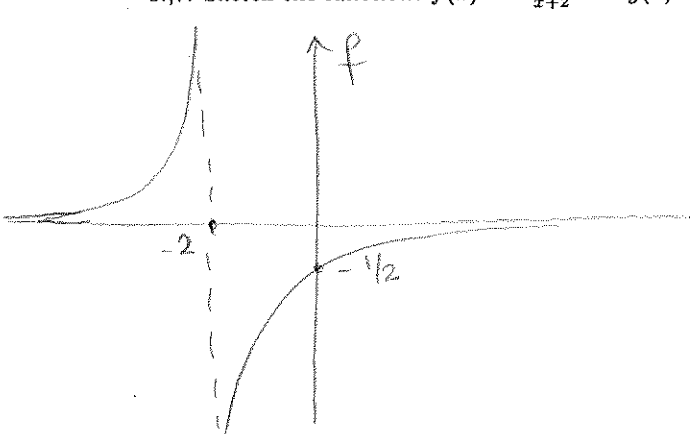
pt-slope formula $y = y_A + m(x - x_A)$

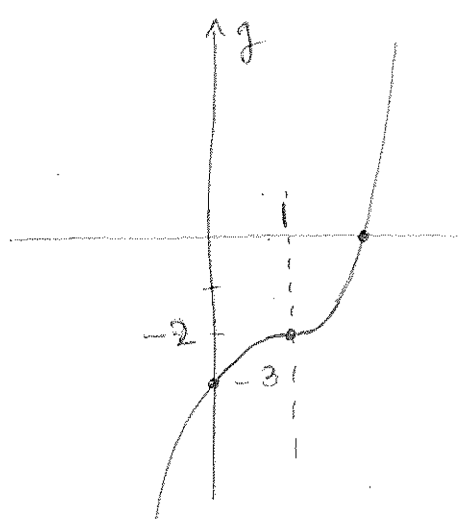
$= 2 - 2(x-1) = 2 - 2x + 2 = 4 - 2x$

Given the functions $f(x) = \frac{1}{x}$ and $g(x) = \ln(x)$

2. What is the domain of f? $(-\infty, 0) \cup (0, +\infty)$
3. Simplify $f \circ f(x)$? $f(f(x)) = 1/f(x) = 1/(1/x) = x$
4. Simplify $g \circ f(x)$? $\ln(1/x) = -\ln x$
5. Is the function $f(x)$ odd, even, or neither? odd

6.,7. Sketch the functions $f(x) = -\frac{1}{x+2}$ and $g(x) = (x-1)^3 - 2$





8. Complete the square for the expression $3x^2 + x - 1$:

$$3\left(x^2 + \frac{1}{3}x + \frac{1}{36} - \frac{1}{36}\right) - 1 = 3\left(\left(x+\frac{1}{6}\right)^2 - \frac{1}{36}\right) - 1$$

$$= 3\left(x+\frac{1}{6}\right)^2 - \frac{1}{12} - 1 = 3\left(x+\frac{1}{6}\right)^2 - \frac{13}{12}$$

Given the parabola $y = 2x^2 + x - 5$:

9. What is the x-coordinate of the vertex? $-b/2a = -1/2(2) = -1/4$

10. Does it open up or down? Up

11. What is the y–intercept? -5

12. What are the x–intercepts?

$$D = b^2 - 4ac = (1)^2 - 4(2)(-5) = 1 + 40 = 41$$

$$x_{1,2} = \frac{-1 \pm \sqrt{41}}{4}$$

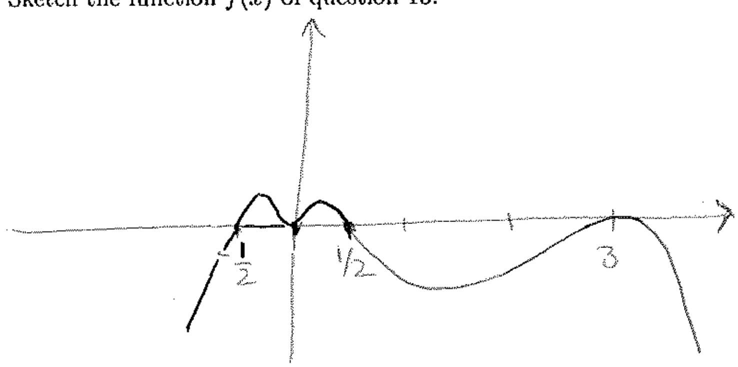
13. Draw a signs table for the function $f(x) = (x^2 - 4x^4)(x^2 - 6x + 9)$

$$f(x) = x^2(1-4x^2)(x^2-6x+9) = x^2(1-2x)(1+2x)(x-3)^2$$

$0, \frac{1}{2}, -\frac{1}{2}, 3$

		$-1/2$		0		$1/2$		3	
x^2	+		+	0	+		+		+
$1-2x$	+		+		+	0	−		−
$1+2x$	−	0	+		+		+		+
$(x-3)^2$	+		+		+		+	0	+
	−	0	+	0	+	0	−	0	−

14. Sketch the function $f(x)$ of question 13.



15. Given the function $f(x)$ of question 13, what is the domain of $\ln(f(x))$?

$$(-\infty, -1/2) \cup (0, 1/2)$$

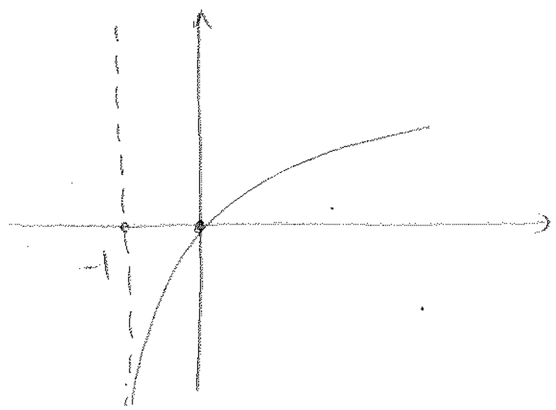
16. Simplify $f(x) = \frac{9^x 3^{-x}}{3^{3x} 27^{-x}}$ $= \frac{(3^2)^x \, 3^{-x}}{3^{3x} \, (3^3)^{-x}} = \frac{3^{2x} \, 3^{-x}}{3^{3x} \, 3^{-3x}} = 3^{2x-x-3x+3x}$

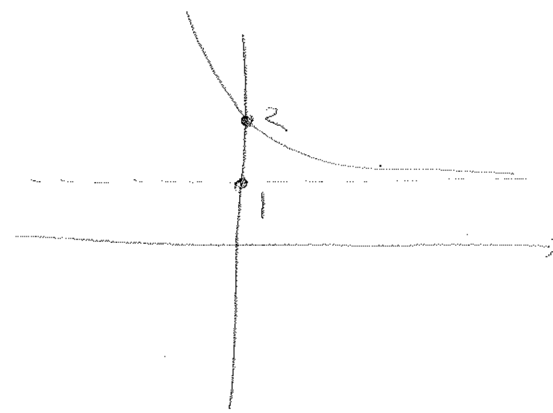
$= 3^x$

17. Simplify $\log_4(16x^2) + 2\log_4(1/x)$

$\log_4(16x^2) + \log_4\left(\frac{1}{x^2}\right) = \log_4\left(16x^2 \cdot \frac{1}{x^2}\right) = \log_4(16) = 2$

18. 19. Sketch the functions $\ln(x+1)$ and $e^{-x}+1$.





19. Expand the function $f(x) = \ln\left[\frac{2e^x(x-1)^3(x-2)}{(4x-1)^5(x+3)}\right]$ into a sum (or difference) of logarithms.

$\ln(2) + \ln(e^x) + \ln((x-1)^3) + \ln(x-2) - \ln((4x-1)^5) - \ln(x+3).$

$= \ln 2 + x + 3\ln(x-1) + \ln(x-2) - 5\ln(4x-1) - \ln(x+3)$

21. Simplify $\log_3(e^{2x})$

$= \frac{\ln(e^{2x})}{\ln 3} = \frac{2x}{\ln 3}$

22. Solve the equation $x^{2\pi} - 2x^\pi + 1 = 0$ for x

let $u = x^\pi$ then $u^2 - 2u + 1 = 0 \Rightarrow (u-1)^2 = 0 \Rightarrow u = 1$

so $x^\pi = 1$ so $x = 1^{1/\pi} = 1$

23. Solve the equation $x^4 + 3x^2 + 2 = 0$ for x

Let $u = x^2$ then $u^2 + 3u + 2 = 0$

then $(u+1)(u+2) = 0$ so $u = -1$ or $u = -2$

$u = -1$: $x^2 = -1 \rightarrow$ no solution

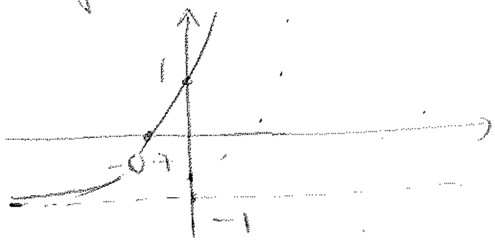
$u = -2$: $x^2 = -2 \rightarrow$ no solution

$\rightarrow$ no solution to this equation!

24.,25. Calculate the x- and y-intercepts of the function $f(x) = 2e^x - 1$, then sketch the function. Hint: $\ln(2) \simeq 0.7$.

y-intercept: $f(0) = 2e^0 - 1 = 2 - 1 = 1$

x-intercept: $2e^x - 1 = 0 \quad . \quad 2e^x = 1 \quad e^x = 1/2 \quad x = \ln(1/2)$
$= -\ln 2 = -0.7$



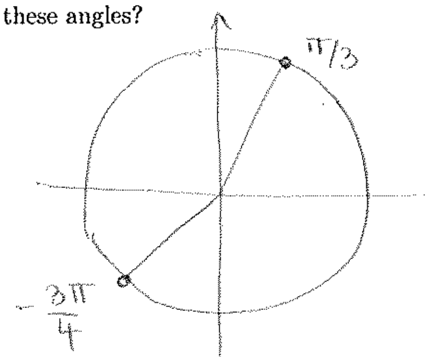
26. Solve the equation $4^x = 5^{1/x}$

$\ln(4^x) = \ln(5^{1/x}) \qquad x \ln 4 = \frac{1}{x} \ln 5$

$x^2 = \frac{\ln 5}{\ln 4} \qquad x = \pm\sqrt{\frac{\ln 5}{\ln 4}}$

27. $\log_4(2x^3) = 3\log_4(2x)$: TRUE/FALSE

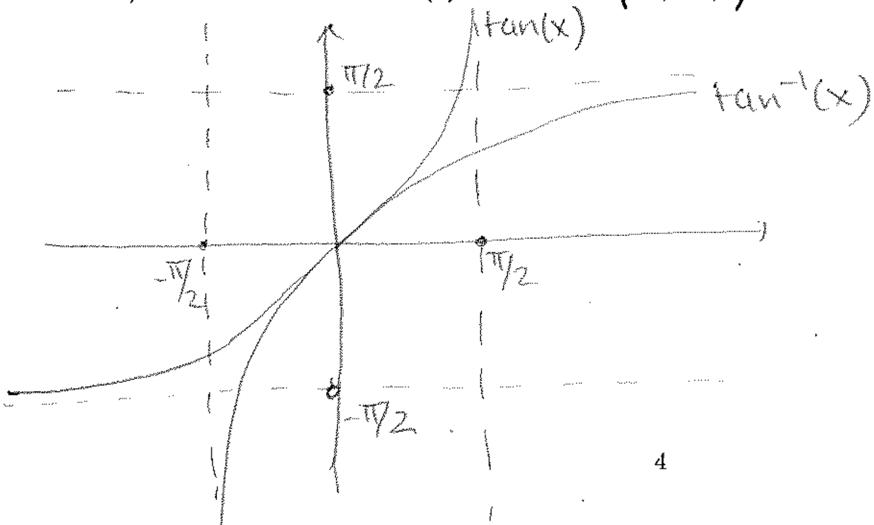
28.,29.,30.,31. Draw a unit circle, and place the angles $\pi/3$, and $-3\pi/4$. What are the tangents of these angles?



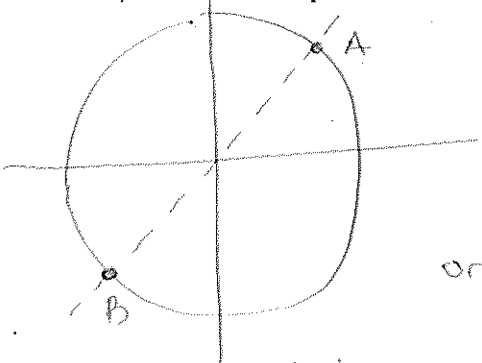
$\tan\frac{\pi}{3} = \frac{\sin \pi/3}{\cos \pi/3} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$

$\tan\left(-\frac{3\pi}{4}\right) = \frac{\sin(-3\pi/4)}{\cos(-3\pi/4)} = \frac{-\sqrt{2}/2}{-\sqrt{2}/2} = 1$

32.,33. Sketch the function $\tan(x)$ on the interval $(-\pi/2, \pi/2)$ and use this to sketch the graph of $\tan^{-1}(x)$.



34.,35. Draw all the points on the unit circle where $\tan(x) = 1$, and solve this equation for x. .



$\tan(x) = 1 \Rightarrow \sin(x) = \cos(x)$

$\Rightarrow y = x$

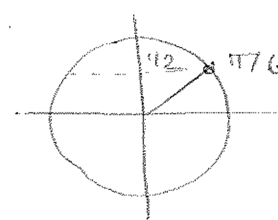
A: $x = \frac{\pi}{4} + 2k\pi$

or B. $x = -\frac{3\pi}{4} + 2k\pi$.

36. Show that $\cos(\sin^{-1}(x)) = \pm\sqrt{1-x^2}$ using the Pythagorean formula

$\cos(x) = \pm\sqrt{1 - \sin^2(x)}$ so

$\cos(\sin^{-1}(x)) = \pm\sqrt{1 - \sin^2(\sin^{-1}(x))} = \pm\sqrt{1 - x^2}$



37. What is $\sin^{-1}(1/2)$? $\pi/6$

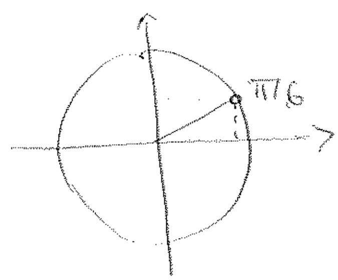
38. Use the formula $\cos(a - b) = \cos a \cos b + \sin a \sin b$ to calculate $\cos(\pi/12)$.
Hint: write $\pi/12$ as $\pi/4 - \pi/6$.

$\cos\left(\frac{\pi}{12}\right) = \cos\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$

$= \cos\frac{\pi}{4}\cos\frac{\pi}{6} + \sin\frac{\pi}{4}\sin\frac{\pi}{6}$

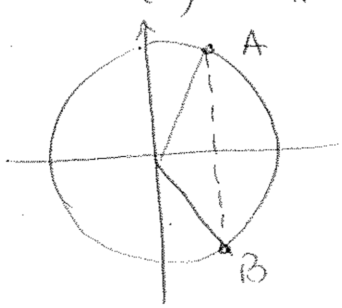
$= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \frac{\sqrt{2}}{2}\left(\frac{\sqrt{3}}{2} + \frac{1}{2}\right)$

$= \frac{\sqrt{2}}{4}(1 + \sqrt{3})$



39.,40. Simplify the equation $\cos^2(x) - \sin^2(x) = \frac{1}{2}$, and solve the equation for x.

$\cos^2(x) - \sin^2(x) = \cos 2x = \frac{1}{2}$



A: $2x = \frac{\pi}{3} + 2k\pi$ so $x = \frac{\pi}{6} + k\pi$

B: $2x = -\frac{\pi}{3} + 2k\pi$ so $x = -\frac{\pi}{6} + k\pi$

$$f(x) = \frac{2(x+1)(x-5/2)}{4(x+1)(x+3/4)} = \frac{2(x-5/2)}{4(x+3/4)} = \frac{x-5/2}{2(x+3/4)} \quad \text{if } x \neq -1$$

PROBLEM 2: RATIONAL FUNCTIONS.[15 POINTS] Consider the function $f(x) = \frac{2x^2-3x-5}{4x^2+7x+3}$.

(a) Factor and simplify this expression.

$2x^2-3x-5$: $D = (-3)^2 - 4(2)(-5) = 9 + 40 = 49$

$x_{1,2} = \frac{-(-3) \pm \sqrt{49}}{2(2)} = \frac{3 \pm 7}{4} = \begin{cases} 10/4 = 5/2 \\ -1 \end{cases}$

$2x^2-3x-5 = 2(x+1)(x-5/2)$

$4x^2+7x+3$: $D = (7)^2 - 4(4)(3) = 49 - 48 = 1$

$x_{1,2} = \frac{-7 \pm \sqrt{1}}{2(4)} = \frac{-7 \pm 1}{8} = \begin{cases} -1 \\ -3/4 \end{cases}$

$4x^2+7x+3 = 4(x+1)(x+3/4)$

(b) What is the domain of $f(x)$? $\rightarrow \mathbb{R} - \{-1, -3/4\}$

(c) What is the y-intercept? $-5/3$

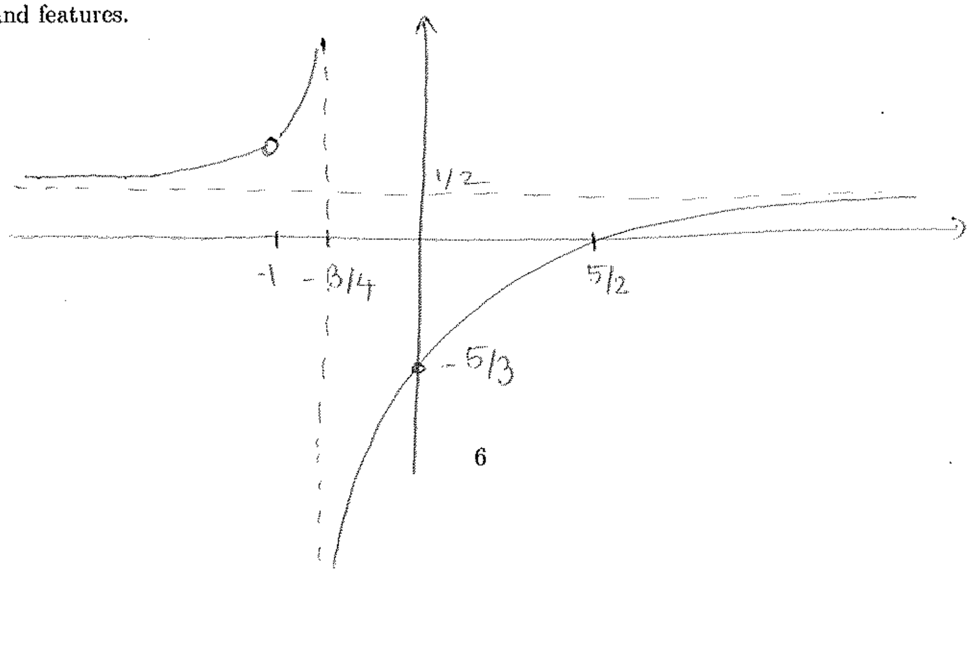
(d) What are the x-intercepts? $5/2$

(e) Draw a signs table for $f(x)$

		$-3/4$		$5/2$	
$x-5/2$	$-$		$-$	0	$+$
$x+3/4$	$-$	∞	$+$		$+$
	$+$	∞	$-$	0	$+$

(e) What is the behavior of $f(x)$ as x goes to $+\infty$ and $-\infty$? $f(x)$ tends to $\frac{1}{2}$ (horizontal asymptote)

(f) Using this information, sketch $f(x)$, making sure to annotate your graph with all the important points and features.



PROBLEM 3. LOGARITHMS AND EXPONENTIALS. [15 POINTS]

(a) Consider the logarithm in base 10. What is:

- $\log_{10}(0.01) =$ ___-2___, $\log_{10}(0.1) =$ ___-1___, $\log_{10}(1) =$ ___0___.
- $\log_{10}(10) =$ ___1___, $\log_{10}(100) =$ ___2___, $\log_{10}(1000) =$ ___3___.

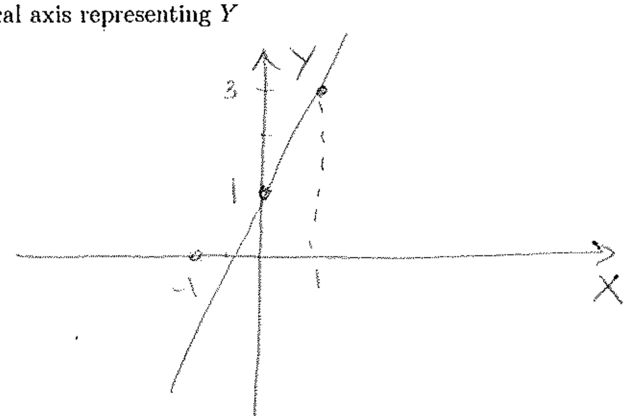
(b) Given $y = 10x^2$, what is $\log_{10}(y)$? Simplify your expression.

$$\log_{10}(10x^2) = \log_{10}(10) + \log_{10}(x^2)$$
$$= 1 + 2\log_{10}(x)$$

(c) Suppose we create two new variables, $Y = \log_{10}(y)$ and $X = \log_{10}(x)$. What is the function f that relates Y to X as $Y = f(X)$?

$$Y = \log_{10}(y) = \log_{10}(10x^2) = 1 + 2\log_{10}(x) = 1 + 2X$$

(d) Sketch the function $Y = f(X)$ you just found on a graph with horizontal axis representing X, and vertical axis representing Y



(e) Generally speaking, consider a power-law relationship of the kind $y = ax^b$. If $Y = \log_{10}(y)$ and $X = \log_{10}(x)$, what is the relationship between X and Y?

$$Y = \log_{10}(y) = \log_{10}(ax^b) = \log_{10}(a) + \log_{10}(x^b)$$
$$= \log_{10}(a) + b\log_{10}(x) = \log_{10}(a) + bX$$

(f) This shows that the graph of a power law $y = ax^b$ on a graphing paper which has $Y = \log_{10}(y)$ on the vertical axis, and $X = \log_{10}(x)$ on the horizontal axis (this is called log-log paper) always looks like a straight line.

- The slope of that straight line is ___b___
- The y intercept of that straight line is ___$\log_{10}(a)$___

PROBLEM 4. TRIGONOMETRIC FUNCTIONS. [15 POINTS]

(a) Using the double-angle formulas, show that $\cos(2x) + \frac{\sin(2x)}{\sin(x)} = 2\cos^2(x) + 2\cos(x) - 1$

$$\cos 2x + \frac{\sin 2x}{\sin x} = \cos^2 x - \sin^2 x + \frac{2\sin x \cos x}{\sin x}$$
$$= \cos^2 x - \sin^2 x + 2\cos x$$
$$= \cos^2 x - (1 - \cos^2 x) + 2\cos x$$
$$= \cos^2 x - 1 + \cos^2 x + 2\cos x$$
$$= 2\cos^2 x + 2\cos x - 1 \quad \checkmark$$

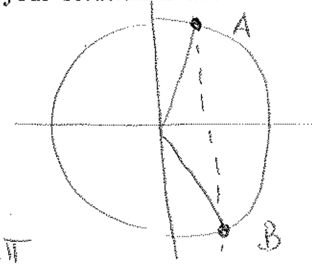
(b) Let's define the new variable $u = \cos(x)$. Transform the equation $2\cos^2(x) + 2\cos(x) - 1 = 0$ into an equation for u, then solve it for u. Hint: $\sqrt{12} \simeq 3.6$.

$$\Rightarrow \quad 2u^2 + 2u - 1 = 0 \qquad D = (2)^2 - 4(2)(-1) = 4 + 8 = 12$$

$$u_{1,2} = \frac{-2 \pm \sqrt{12}}{2(2)} = \frac{-2 \pm \sqrt{12}}{4} \simeq \begin{cases} \frac{-2+3.6}{4} = \frac{1.6}{4} \simeq 0.4 \\ \frac{-2-3.6}{4} \simeq \frac{-5.6}{4} < -1 \end{cases}$$

$$\to \quad u_1 = \frac{-2+\sqrt{12}}{4} \qquad u_2 = \frac{-2-\sqrt{12}}{4}$$

(c) Now solve the equation $u = \cos(x)$ for x where u is the first, then the second solution found above. Make sure to find *all* of the solutions for x, and express your solution if needed in terms of inverse trigonometric functions.



$\cos^{-1} x$ returns a number between 0 and π

$\to$ returns A.

But angle B is $-$angle A.

Case of u_1:

$$\cos x = \frac{-2+\sqrt{12}}{4} \simeq 0.4$$

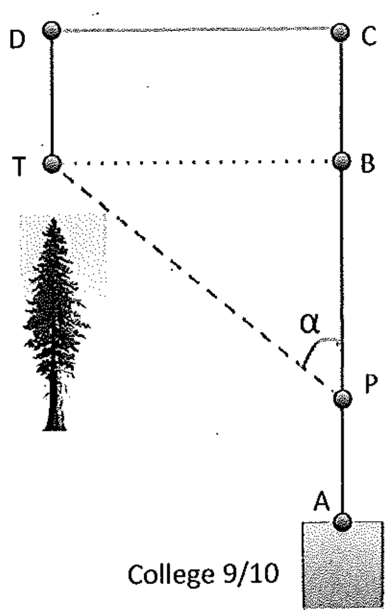
$$A: \quad x = \cos^{-1}\left(\frac{-2+\sqrt{12}}{4}\right) + 2k\pi$$

$$B: \quad x = -\left[\cos^{-1}\left(\frac{-2+\sqrt{12}}{4}\right) + 2k\pi\right] = -\cos^{-1}\left(\frac{-2+\sqrt{12}}{4}\right) + 2k'\pi$$

Case of u_2: $\quad \cos x = u_2$ does not have solutions because $u_2 < -1$.

Applied Problem. [15 points]

A group of students would like to go from College 9/10 to Tree Nine on upper campus, as quickly as possible. They are given the following map. They can walk on the main trails (solid lines) at 2 miles per hour, but are slowed down to 0.5 mile per hour when they cut through the underbrush (dashed line). Tree Nine is sufficiently tall that they can see it from the path, and they can decide to cut anywhere (at the point P). When they do, they can measure the angle between the road and their new path. This angle is called α. The points A, B, C, D and T are therefore fixed, but the point P can vary depending on the decision made by the group. (Note: the distances have been exaggerated to make the numbers easier to deal with. Tree Nine is not actually that far from College 9/10).



AC = 1 mile CD = 0.5 miles DT = 0.25 mil

BC = ...0.25...

BT = ...0.5...

For the next two questions, write your answer as the ratio of two lengths

$\tan\alpha = \ldots BT/BP$ $\sin\alpha = \ldots BT/PT$

For the following 3 lengths, the answer depends on α (which is a known variable).

PB = ... $BT/\tan\alpha = 0.5/\tan\alpha$

PA = ...AC ... $- BC - BP = 1 - 0.25 - \frac{0.5}{\tan\alpha} = 0.75 - \frac{0.5}{\tan\alpha}$

PT = ... $BT/\sin\alpha = 0.5/\sin\alpha$

(a) Fill in the gaps above for each of the lengths.

(b) How long does it take to walk to the tree when only following the road (A to C to D to T) ?

$$\text{time} = \frac{\text{distance}}{\text{velocity}} = \frac{\text{distance AC} + \text{distance CD} + \text{distance DT}}{2}$$

$$= \frac{1+0.5+0.25}{2} = \frac{1.75}{2} = \frac{1}{2} + \frac{0.75}{2} = 0.5 + 0.375$$
$$= 0.875$$

(c) Using the underbrush shortcut (A to P then P to T), how long does it take to walk to the tree (circle the right answer, and justify your answer on the next page)

- $1.5 - \frac{1}{\tan\alpha} + \frac{0.25}{\sin\alpha}$ hours
- $0.375 - \frac{0.25}{\tan\alpha} + \frac{1}{\sin\alpha}$ hours
- , $\tan\alpha - \frac{0.5}{\sin\alpha}$ hours

1.375 +

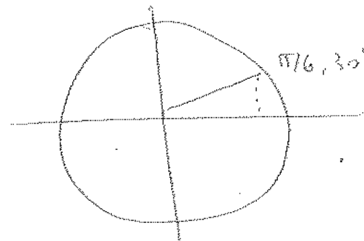
Justification:
$$\frac{\text{distance } AP}{2 \text{ mph}} + \frac{\text{distance } PT}{0.5 \text{ mph}}$$

$$= \frac{0.75 - \frac{0.5}{\tan\alpha}}{2} + \frac{\cancel{0.5}}{\sin\alpha} \cdot \frac{1}{\cancel{0.5}}$$

$$= \frac{0.75}{2} - \frac{1}{4\tan\alpha} + \frac{1}{\sin\alpha} = 0.375 - \frac{0.25}{\tan\alpha} + \frac{1}{\sin\alpha}.$$

(d) How long does the trip take if they make an angle $\alpha = 30°$? Hint: . $\sqrt{3} \cong 1.6$

$$0.375 - \frac{0.25}{\tan(30°)} + \frac{1}{\sin(30°)}$$



$\sin 30° = \frac{1}{2}$

$\cos 30° = \sqrt{3}/2$

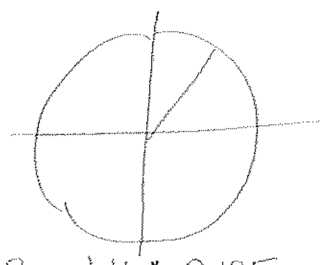
$\tan 30° = \frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}}$

$$= 0.375 - \frac{0.25}{\frac{1}{\sqrt{3}}} + \frac{1}{\frac{1}{2}}$$

$$= 0.375 - \frac{\sqrt{3}}{4} + 2 = 0.375 - 0.4 + 2 = 1.6 + 0.375 = 1.975$$

(e) How long does the trip take if they make an angle $\alpha = 45°$? Hint: $1/\sqrt{2} \simeq 0.7$,

$$0.375 - \frac{0.25}{\tan 45°} + \frac{1}{\sin 45°}$$



$\sin 45° = \frac{\sqrt{2}}{2}$

$\cos 45° = \frac{\sqrt{2}}{2}$

$\tan 45° = 1$

$$= 0.375 - 0.25 + \frac{1}{\sqrt{2}/2}$$

$$= 0.375 - 0.25 + \frac{2}{\sqrt{2}} = 0.375 - 0.25 + 0.7 \times 2 = 1.4 + 0.125 = 1.525$$

(f) What is the angle α if they decide to go straight from point A to point T (i.e. if the point P is the same as the point A)? Express your answer in terms of an inverse trigonometric function. Hint: first calculate the distance AB.

$AB = AC - BC = 1 - 0.25 = 0.75$

If P is at A then

$PB = AB = \frac{0.5}{\tan\alpha} = 0.75$ so $\tan\alpha = \frac{0.5}{0.75} = \frac{1/2}{3/4} = \frac{2}{3}$

then $\alpha = \tan^{-1}(2/3)$

se.