Practice Final 1, MATH 3

Name: ______________________________

Calculators are not allowed.

Read all the questions before you start working on any of them. Start with the ones you are most comfortable with, and continue with the other ones later. Always double-check your answers.

Relax, and do your best!

PROBLEM 1: SHORT QUESTIONS. [40 POINTS]

1. What is the equation of the line which goes through the points $\overset{A}{(3,2)}$ and $\overset{B}{(1,1)}$?

slope : $m = \frac{y_B - y_A}{x_B - x_A} = \frac{1-2}{1-3} = \frac{-1}{-2} = \frac{1}{2}$

$$y = y_A + m(x - x_A) = 2 + \frac{1}{2}(x-3) = \frac{1}{2}x + 2 - \frac{3}{2} = \frac{1}{2}x + \frac{1}{2}$$

Given the functions $f(x) = e^x$ and $g(x) = \ln(x-2)$

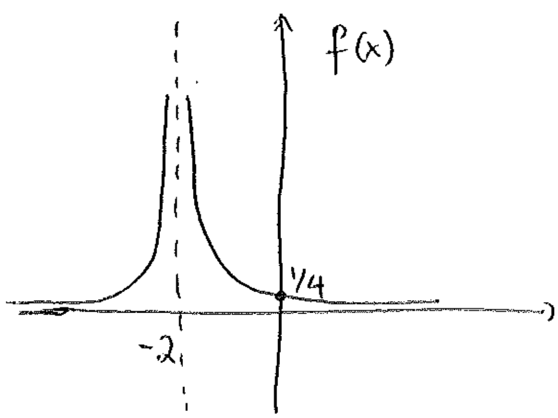
2. What is the domain of g? $x > 2$, $(2, +\infty)$

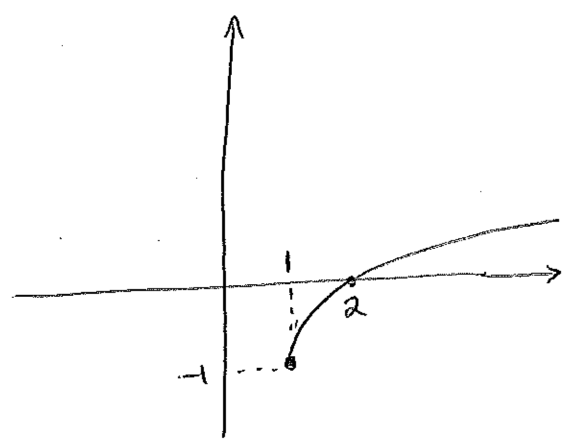
3. What is $f \circ g(x)$? $f(g(x)) = e^{g(x)} = e^{\ln(x-2)} = x-2$

4. What is $g \circ f(x)$? $g(f(x)) = \ln(f(x)-2) = \ln(e^x - 2)$

5. Given the function $f(x) = \ln(\sqrt{x})$ for $x > 0$, what is $f^{-1}[f(x^4+1)]$? x^4+1

6.,7. Sketch the functions $f(x) = \frac{1}{(x+2)^2}$ and $g(x) = \sqrt{x-1} - 1$





8. Complete the square for the expression $2x^2 - 2x + 1$:

$$2(x^2 - x) + 1 = 2\left(x^2 - x + \frac{1}{4} - \frac{1}{4}\right) + 1$$

$$= 2\left(\left(x - \frac{1}{2}\right)^2 - \frac{1}{4}\right) + 1 = 2\left(x - \frac{1}{2}\right)^2 - \frac{1}{2} + 1 = 2\left(x - \frac{1}{2}\right)^2 + \frac{1}{2}$$

Given the parabola $y = x^2 - 2x - 6$:

9. What are the coordinates of the vertex? $x_v = -b/2a = -(-2)/2(1) = 1$ $\qquad y_v = (1)^2 - 2(1) - 6 = -7$

10. Does it open up or down? up

11. What is the y-intercept? -6

12. What are the x-intercepts?

$$D = b^2 - 4ac = (-2)^2 - 4(1)(-6) = 4 + 24 = 28$$

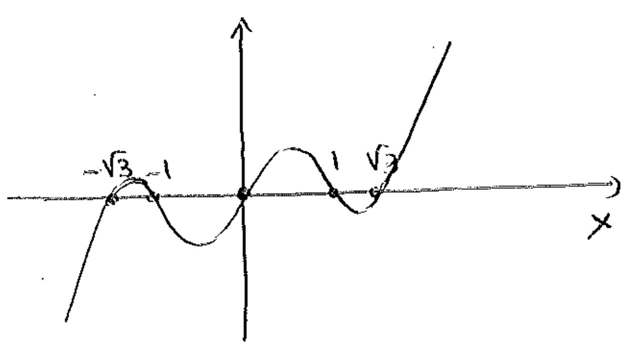
$$x_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = \frac{-(-2) \pm \sqrt{28}}{2} = \frac{2 \pm \sqrt{28}}{2} = 1 \pm \sqrt{7}$$

13. Draw a signs table for the function $f(x) = (x^3 - x)(x^2 - 3)$

$$(x^3 - x)(x^2 - 3) = x(x^2 - 1)(x^2 - 3) = x(x-1)(x+1)(x - \sqrt{3})(x + \sqrt{3})$$

		$-\sqrt{3}$		-1		0		1		$\sqrt{3}$	
x	$-$		$-$		$-$	0	$+$		$+$		$+$
$x-1$	$-$		$-$		$-$		$-$	0	$+$		$+$
$x+1$	$-$		$-$	0	$+$		$+$		$+$		$+$
$x-\sqrt{3}$	$-$		$-$		$-$		$-$		$-$	0	$+$
$x+\sqrt{3}$	$-$	0	$+$		$+$		$+$		$+$		$+$
	$-$	0	$+$	0	$-$	0	$+$	0	$-$	0	$+$

14. Sketch the function $f(x)$ of question 13.



15. Given the function $f(x)$ of question 13, what is the domain of $\ln(f(x))$? $(-\sqrt{3}, -1) \cup (0, 1) \cup (\sqrt{3}, +\infty)$

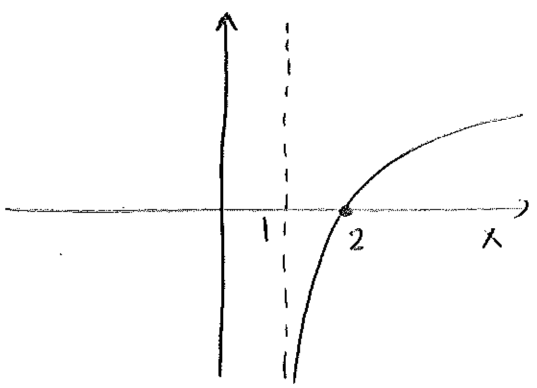
16. Simplify $f(x) = \frac{2^x 4^{-2x}}{8^{3x} 2^{-x}}$ $= \frac{2^x (2^2)^{-2x}}{(2^3)^{3x}\, 2^{-x}} = \frac{2^x\, 2^{-4x}}{2^{9x}\, 2^{-x}} = 2^{x-4x-9x+x}$

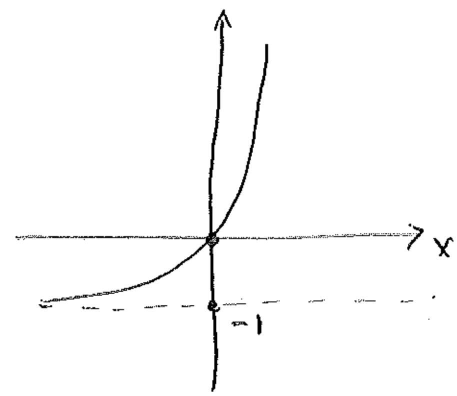
$= 2^{-11x} = \frac{1}{2^{11x}}$

17. Simplify $\log_4(2x^2) - 2\log_4(x)$

$= \log_4(2x^2) - \log_4(x^2) = \log_4\left(\frac{2x^2}{x^2}\right) = \log_4(2) = \frac{1}{2}$

18. 19. Sketch the functions $\log_2(x-1)$ and $2^x - 1$.





19. Expand the function $f(x) = \ln\left[\frac{2x^2(x-1)(x+1)}{4x^3}\right]$ into a sum (or difference) of logarithms.

$\ln(2x^2) + \ln(x-1) + \ln(x+1) - \ln(4x^3)$

$= \ln 2 + 2\ln x + \ln(x-1) + \ln(x+1) - \ln 4 - 3\ln x$ ← ok here

$= \ln(1/2) - \ln x + \ln(x-1) + \ln(x+1)$

21. Simplify $\log_2(e^x)$

$\log_2(e^x) = \frac{\ln(e^x)}{\ln 2} = \frac{x}{\ln 2}$

22. What is the inverse of the function $f(x) = 4x^{-1/4}$?

$y = 4x^{-1/4}$ $\quad x^{-1/4} = \frac{y}{4}$ $\quad x^{1/4} = \frac{4}{y}$ $\quad x = \left(\frac{4}{y}\right)^4$

So $f^{-1}(x) = \left(\frac{4}{x}\right)^4$

23. What is the inverse of the function $f(x) = 2^{x+1} - 3$

$y = 2^{x+1} - 3$ $\quad y + 3 = 2^{x+1}$ $\quad \ln(y+3) = (x+1)\ln 2$

$(x+1) = \frac{\ln(y+3)}{\ln 2}$ $\quad x = \frac{\ln(y+3)}{\ln 2} - 1$

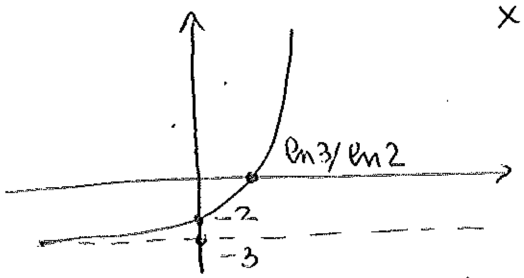
$f^{-1}(x) = \frac{\ln(x+3)}{\ln 2} - 1$

24.,25. Calculate the x- and y-intercepts of the function $f(x) = 2^x - 3$, then sketch the function.

y-intercept: $f(0) = 2^0 - 3 = 1 - 3 = -2$

x-intercept: $2^x - 3 = 0 \quad 2^x = 3 \quad \ln(2^x) = \ln 3$

$x \ln 2 = \ln 3 \quad x = \ln 3 / \ln 2$



26. Solve the equation $2^x = 3^{x-1}$

$\ln(2^x) = \ln(3^{x-1}) \Rightarrow x \ln 2 = (x-1)\ln 3 = x\ln 3 - \ln 3$

$\Rightarrow x\ln 2 - x\ln 3 = -\ln 3$

$\Rightarrow x(\ln 2 - \ln 3) = -\ln 3 \Rightarrow x = \dfrac{\ln 3}{\ln 3 - \ln 2}$

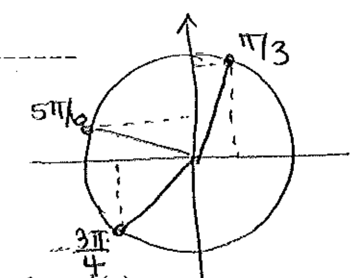
27. $\log_4(x^3) = \frac{3}{\ln(4)}\ln(x)$: TRUE / FALSE

28. Write $2^{x/a}$ as a natural exponential. $e^{\frac{x}{a}\ln 2}$

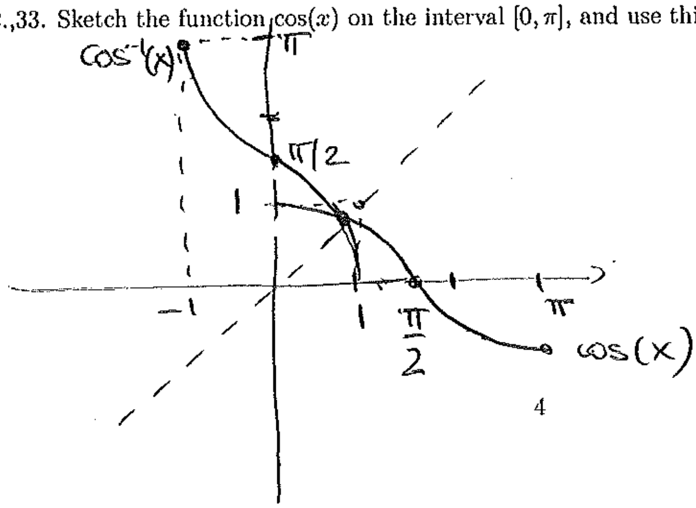
29. What is $\tan(\pi/3)$? $\dfrac{\sin \pi/3}{\cos \pi/3} = \dfrac{\sqrt{3}/2}{1/2} = \sqrt{3}$

30. What is $\sin(5\pi/6)$? $+1/2$

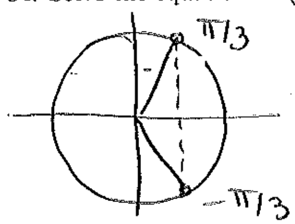
31. What is $\cos(-3\pi/4)$? $-\sqrt{2}/2$



32.,33. Sketch the function $\cos(x)$ on the interval $[0, \pi]$, and use this to sketch the graph of $\cos^{-1}(x)$.



34. Solve the equation $\sec(x) = 2$.



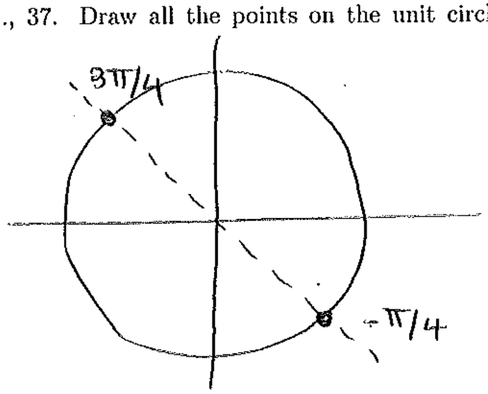
$$\frac{1}{\cos x} = 2 \implies \cos x = \frac{1}{2}$$

$$\implies x = \pi/3 + 2\pi k \quad \text{or} \quad x = -\frac{\pi}{3} + 2\pi k.$$

35. Simplify the expression $\frac{\sin(2x)}{2\sin(x)} - \frac{1}{\sec(x)}$

$$\frac{2\sin(x)\cos(x)}{2\sin(x)} - \frac{1}{\frac{1}{\cos x}} = \cos x - \cos x = 0$$

36., 37. Draw all the points on the unit circle where $\cos(x) = -\sin(x)$, and solve this equation for x.



$$x = -\frac{\pi}{4} + 2\pi k \quad \text{or} \quad x = \frac{3\pi}{4} + 2\pi k$$

(alternatively:

$$x = \frac{3\pi}{4} + \pi k\ )$$

38. What is $\cos(\cos^{-1}(x))$ if $x > 1$? ____ does not exist ____

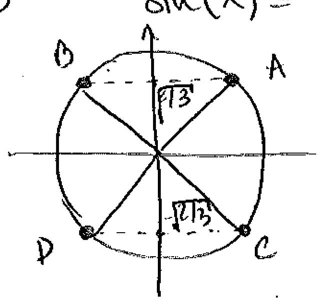
39.,40. Simplify the equation $\cos^2(x) + 4\sin^2(x) = 3$, and solve the equation for x.

$$\cos^2(x) + \sin^2(x) + 3\sin^2(x) = 3$$

$$1 + 3\sin^2(x) = 3 \qquad 3\sin^2(x) = 2$$

$$\sin^2(x) = 2/3 \qquad (\text{also } \cos^2(x) = 1/3)$$

$$\implies \sin(x) = \pm\sqrt{2/3}$$



A: $x = \sin^{-1}(\sqrt{2/3}) + 2k\pi$

C: $x = \sin^{-1}(-\sqrt{2/3}) + 2k\pi$.

B: $x = \pi - \sin^{-1}(\sqrt{2/3}) + 2k\pi$

D: $x = -\pi - \sin^{-1}(-\sqrt{2/3}) + 2k\pi$

PROBLEM 2: RATIONAL FUNCTIONS. [15 POINTS] Consider the function $f(x) = \frac{1}{x-2} + \frac{1}{(x-3)(x+2)}$.

(a) Reduce to the same denominator, simplify then factor this expression.

$$\frac{1}{x-2} + \frac{1}{(x-3)(x+2)} = \frac{(x-3)(x+2)+(x-2)}{(x-3)(x+2)(x-2)}$$

$$= \frac{x^2-3x+2x-6+x-2}{(x-3)(x+2)(x-2)} = \frac{x^2-8}{(x-3)(x+2)(x-2)}$$

$$= \frac{(x-\sqrt{8})(x+\sqrt{8})}{(x-3)(x+2)(x-2)}$$

(b) What is the domain of $f(x)$? $\mathbb{R} - \{3, 2, -2\}$ or $(-\infty,-2) \cup (-2,2) \cup (2,3) \cup (3,+\infty)$

(c) What is the y–intercept? $-8/12 = -2/3$

(d) What are the x–intercepts? $\sqrt{8}$ and $-\sqrt{8}$

(e) Draw a signs table for $f(x)$

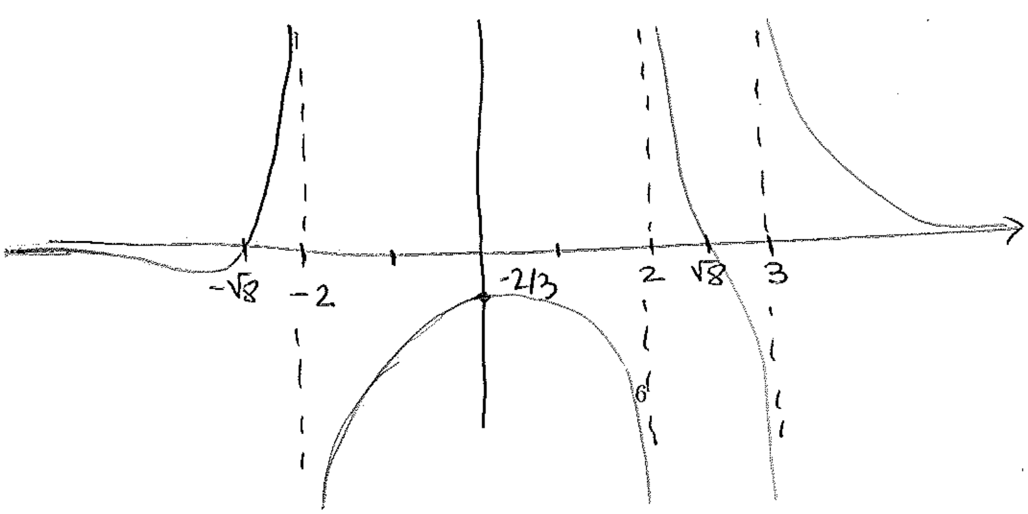
$2 < \sqrt{8} < 3$

		$-\sqrt{8}$		-2		2		$\sqrt{8}$		3	
$x-\sqrt{8}$	−		−		−		−	0	+		+
$x+\sqrt{8}$	−	0	+		+		+		+		+
$x-2$	−		−		−	0	+		+		+
$x+2$	−		−	0	+		+		+		+
$x-3$	−		−		−		−		−	0	+
	−	0	+	∞	−	∞	+	0	−	∞	+

(e) What is the behavior of $f(x)$ as x goes to $+\infty$ and $-\infty$?

$f(x) \approx \frac{1}{x}$ so $f(x)$ goes to 0 in both cases

(f) Using this information, sketch $f(x)$.



PROBLEM 3. LOGARITHMS AND EXPONENTIALS. [15 POINTS]

We consider here the two functions

$$f(t) = 2 \times 2^t \text{ and } g(t) = 2 \times e^{-t\ln 2}$$

where t is measured in hours.

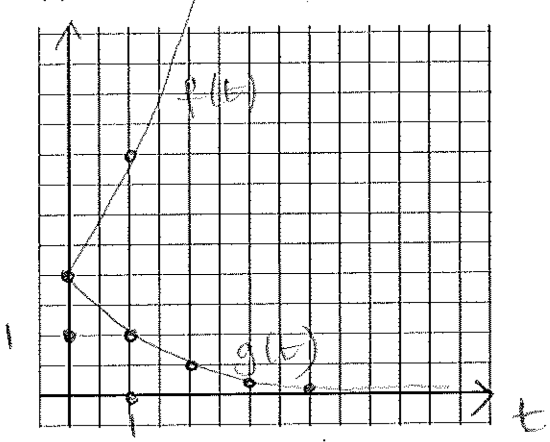
(1) Write $g(t)$ in the form of an exponential in base 2: $g(t) = 2e^{-t\ln 2} = 2e^{t\ln(\frac{1}{2})} = 2\left(\frac{1}{2}\right)^t$

(2) Calculate the following values:

$f(0) =$ 2, $f(1) =$ 4, $f(2) =$ 8, $f(3) =$ 16

$g(0) =$ 2, $g(1) =$ 1, $g(2) =$ 1/2, $g(3) =$ 1/4

(3) Use this information to graph the two functions



(4) Are these two functions inverses of each other? Justify your answer.

No : their graphs are not symmetric with respect to $y = x$ line

(5) For what value of t is $f(t) = 20$?

$f(t) = 20 \Rightarrow 2 \cdot 2^t = 20 \quad 2^t = 10 \quad \ln(2^t) = \ln(10)$

$t\ln 2 = \ln 10 \quad t = \dfrac{\ln 10}{\ln 2}$

(6) What real world system could be described by these two functions (clearly mark either $f(t)$ or $g(t)$ next to the correct real problem). Only one statement is correct in each case.

- A radioactive isotope for which you initially have 2 micrograms, with a half-life of 1 hour $g(t)$
- A population of initially 2 bacteria, which doubles every hour $f(t)$
- ~~A radioactive isotope for which you initially have 2 micrograms, with a half-life of ln(2) hours~~
- ~~A population of initially 2 bacteria, which doubles every ln(2) hours~~

PROBLEM 4. TRIGONOMETRIC FUNCTIONS. [15 POINTS]

(a) Use the addition formula $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$ to express $\cos(3x)$ in terms of cosines and sines of x and $2x$.

$$\cos(3x) = \cos(2x + x) = \cos 2x \cos x - \sin 2x \sin x$$

(b) Using the expression that you found, now use the two double-angle formulas and the Pythagorean formula to prove the identity

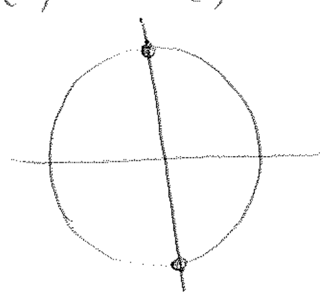
$$\cos(3x) = 4\cos^3(x) - 3\cos(x)$$

$$\begin{aligned}
\cos 3x &= \cos 2x \cos x - \sin 2x \sin x \\
&= (\cos^2 x - \sin^2 x)\cos x - 2\sin x \cos x \sin x \\
&= \cos^3 x - \sin^2 x \cos x - 2\sin^2 x \cos x = \cos^3 x - 3\sin^2 x \cos x \\
&= \cos^3 x - 3(1 - \cos^2 x)\cos x \\
&= \cos^3 x - 3\cos x + 3\cos^3 x = 4\cos^3 x - 3\cos x
\end{aligned}$$

(c) Solve the equation $4\cos^3(x) - 3\cos(x) = 0$ using any method of your choice, and write out all the possible solutions.

Method 1 (easiest!)

$$4\cos^3(x) - 3\cos(x) = 0 \iff \cos(3x) = 0$$



$$\Rightarrow 3x = \frac{\pi}{2} + 2k\pi \quad \text{or} \quad 3x = -\frac{\pi}{2} + 2k\pi$$

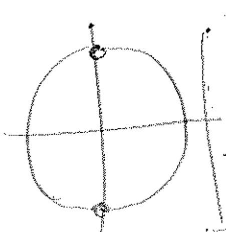
$$\Rightarrow \boxed{x = \frac{\pi}{6} + \frac{2k\pi}{3} \quad \text{or} \quad x = -\frac{\pi}{6} + \frac{2k\pi}{3}}$$

Method 2 (hardest)!

$$\cos x\,[4\cos^2 x - 1] = 0 \iff \cos x\,[2\cos x - 1][2\cos x + 1] = 0$$

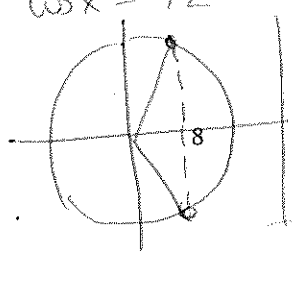
So $\cos x = 0$ or $2\cos x - 1 = 0$ or $2\cos x + 1 = 0$

$\cos x = 0 \Rightarrow$



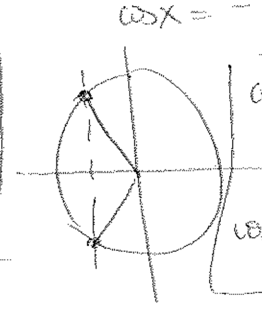
$$\boxed{x = \frac{\pi}{2} + 2k\pi \quad \text{or} \quad x = -\frac{\pi}{2} + 2k\pi}$$

$\cos x = 1/2$



$$\boxed{x = \frac{\pi}{3} + 2k\pi \qquad x = -\frac{\pi}{3} + 2k\pi}$$

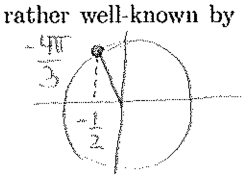
$\cos x = -1/2$



$$\boxed{\cos x = \frac{2\pi}{3} + 2k\pi \qquad \cos x = -\frac{2\pi}{3} + 2k\pi}$$

Applied Problem. [15 points]

Tides are a regular oscillation of the Ocean's mean water height, a phenomenon rather well-known by Santa Cruz surfers. The water height (ignoring waves) varies with time as



$$H(t) = 10 + 1.5\cos\left(\frac{\pi(t-8)}{6}\right)$$

where t is given in hours (t=0 hours being midnight), and H is given in meters.

(a) What is the water height at midnight? $t=0$: $H(0) = 10 + 1.5\cos\left(-\frac{4\pi}{3}\right) = 10 - \frac{1.5}{2}$

$= 9.25$

(b) What is the difference in water height between high-tide (i.e. when the height is maximum) and low-tide (i.e. when the height is minimum)?

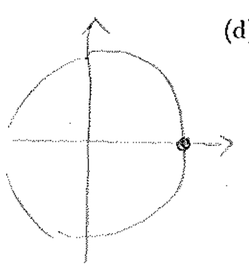
High-tide: $H = 10 + 1.5 = 11.5$

low-tide: $H = 10 - 1.5 = 8.5$

difference: $11.5 - 8.5 = 3$ (= twice amplitude of tide)

(c) What is the period of the tide?

$p = \frac{2\pi}{\frac{\pi}{6}} = 12$ hours



(d) At what times of the day is high-tide?

$H(t) = 11.5 = 10 + 1.5\cos\left(\frac{\pi(t-8)}{6}\right)$

$\Rightarrow \cos\left(\frac{\pi(t-8)}{6}\right) = 1 \Rightarrow \frac{\pi(t-8)}{6} = 0 + 2\pi k$

$t - 8 = \frac{6}{\pi}\cdot 2\pi k = 12k$ so $t = 12k + 8$

(e) Based on this, sketch the function $H(t)$ for t between 0 and 24.

$k=0$ $t = 8$ (8 AM)

$k=1$ $t = 20$ (8PM)