Practice Midterm 2

Name: ________________________________

Section : ________________________________

Calculators are not allowed.

Read all the questions before you start working on any of them. Start with the ones you are most comfortable with, and continue with the other ones later. Always double-check your answers.

Relax, and do your best!

PROBLEM 1: SHORT QUESTIONS. [60 POINTS]

Given the function $f(x) = \frac{x^2-1}{1-x}$

1. What is its domain of definition? ANSWER: $\mathcal{D} = \mathbb{R} - \{1\} = (-\infty, 1) \cup (1, +\infty)$

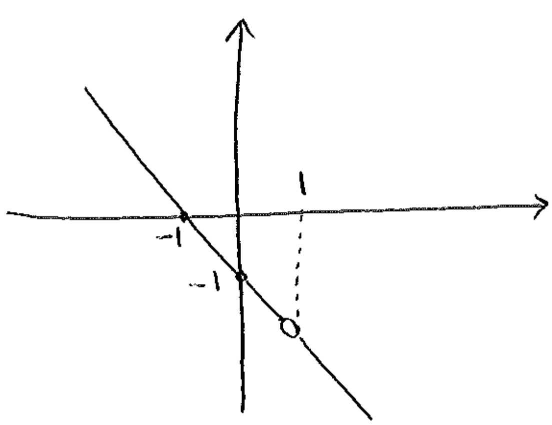
$1 - x \neq 0 \Rightarrow x \neq 1$

2. Simplify $f(x)$, for x inside the domain of definition. ANSWER: $f(x) = -x-1$

$$\frac{x^2-1}{1-x} = \frac{(x-1)(x+1)}{1-x} = -\frac{\cancel{(1-x)}(x+1)}{\cancel{1-x}} \quad \text{if } x \neq 1$$

$$= -(x+1)$$

3. Sketch the function, making sure to annotate your graph.



4. What is the domain of $\ln(f(x))$? ANSWER: $\mathcal{D} = (-\infty, -1)$

$f(x) > 0$ so need $x < -1$

5. Solve the inequality $\frac{5}{x-2} < 1$. ANSWER: $x \in$ $(-\infty, 2) \cup (7, +\infty)$

$$\frac{5}{x-2} - 1 < 0 \rightarrow \frac{5}{x-2} - \frac{x-2}{x-2} < 0 \rightarrow \frac{5-(x-2)}{x-2} < 0$$

$$\rightarrow \frac{5-x+2}{x-2} < 0 \rightarrow \frac{7-x}{x-2} < 0$$

		2		7	
$7-x$	$+$		$+$	0	$-$
$x-2$	$-$	0	$+$		$+$
	$-$	0	$+$	0	$-$

Given the functions $f(x) = e^x$ and $g(x) = \ln(x-2)$

6. What is $f \circ g(x)$? $e^{\ln(x-2)} = x-2$ (Simplify if possible)

7. What is $g \circ f(x)$? $\ln(e^x - 2)$ (Simplify if possible)

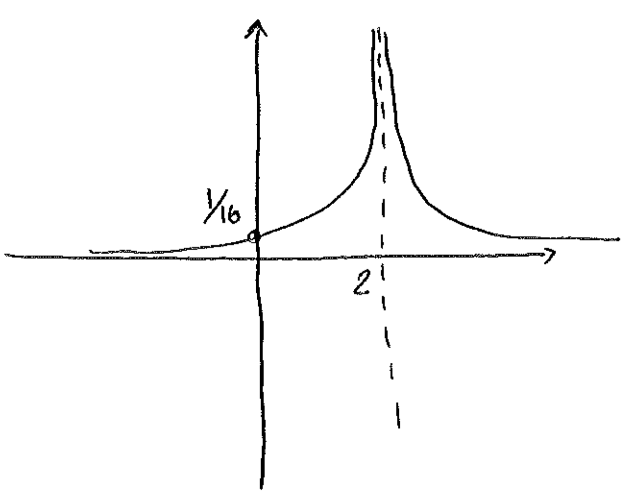
9. Given the function $f(x) = \sqrt{x^2+4}$, what is $f^{-1}(x)$? ANSWER: $f^{-1}(x) =$ $\sqrt{x^2-4}$

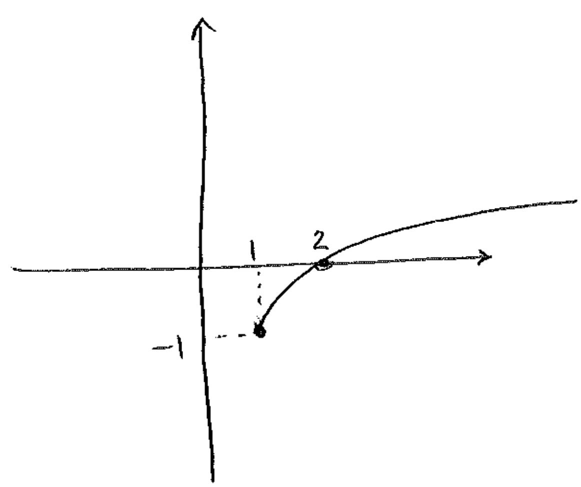
$$y = \sqrt{x^2+4} \Rightarrow y^2 = x^2 + 4 \Rightarrow x^2 = y^2 - 4 \Rightarrow x = \sqrt{y^2-4}$$

$$f^{-1}(y) = \sqrt{y^2-4} \rightarrow f^{-1}(x) = \sqrt{x^2-4}$$

10. Given the function $f(x) = \sqrt{x^2+4}$, what is $f[f^{-1}(x-1)]$? ANSWER: $x-1$

11. 12. Sketch the functions $f(x) = \frac{1}{(x-2)^4}$ and $g(x) = \sqrt{x-1} - 1$, and annotate your graphs





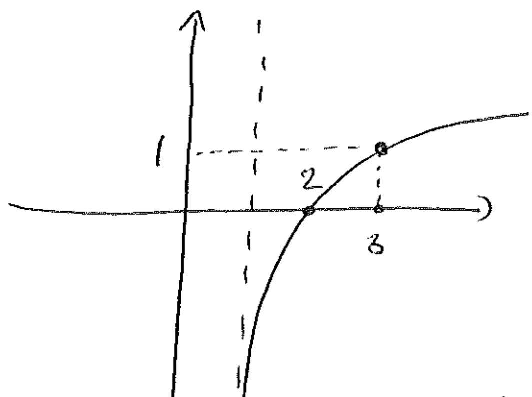
13. Simplify $f(x) = \frac{2^x 4^{-2x}}{8^{3x} 2^{-x}}$. ANSWER: $f(x) =$ 2^{-11x} or $\frac{1}{2^{11x}}$

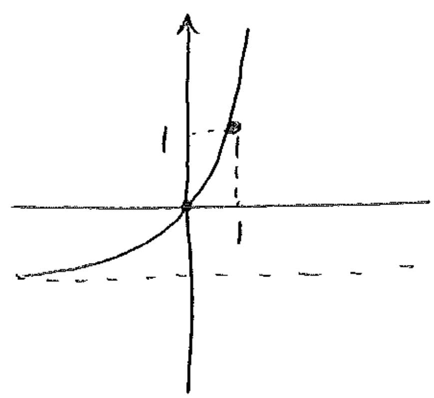
$$\frac{2^x 4^{-2x}}{8^{3x} 2^{-x}} = \frac{2^x (2^2)^{-2x}}{(2^3)^{3x} 2^{-x}} = \frac{2^x 2^{-4x}}{2^{9x} 2^{-x}} = \frac{2^{-3x}}{2^{8x}} = 2^{-3x-8x}$$

$$= 2^{-11x} = \frac{1}{2^{11x}}$$

14. Simplify $\log_4(2x^2) - 2\log_4(x)$. ANSWER: $\log_4\left(\frac{2x^2}{x^2}\right) = \log_4(2)$

15. 16. Sketch the functions $\log_2(x-1)$ and $2^x - 1$, and annotate your graphs.





17. Simplify $\log_2(e^x)$. ANSWER: $x/\ln 2$

$$\log_2(e^x) = \frac{\ln(e^x)}{\ln 2} = \frac{x}{\ln 2}$$

18. Express the function $f(x) = \ln\left(\frac{x^2(x-2)}{(2x-1)^2(x+4)^3}\right)$ as sums and differences of logarithms .

$$\ln(x^2) + \ln(x-2) - \ln((2x-1)^2) - \ln((x+4)^3)$$
$$= 2\ln x + \ln(x-2) - 2\ln(2x-1) - 3\ln(x+4)$$

ANSWER: ________________

19. Solve the equation $2^x = 3^{x-1}$. ANSWER: $\frac{-\ln 3}{\ln 2 - \ln 3} = \frac{\ln(1/3)}{\ln(2/3)}$

$$\ln 2^x = \ln 3^{x-1}$$
$$x\ln 2 = (x-1)\ln 3 \Rightarrow x\ln 2 - x\ln 3 = -\ln 3$$
$$\Rightarrow x(\ln 2 - \ln 3) = -\ln 3 \Rightarrow x = -\ln 3/\ln 2 - \ln 3$$

20. Write $2^{x/a}$ as a natural exponential. ANSWER: ________________

$$2^{x/a} = e^{\frac{x}{a}\ln 2}$$

PROBLEM 2: RATIONAL FUNCTIONS.[20 POINTS] Consider the function $f(x) = \frac{1-x}{x+3}$

(a) What is the x–intercept? $X = 1$

(b) What is the y–intercept? $y = 1/3$

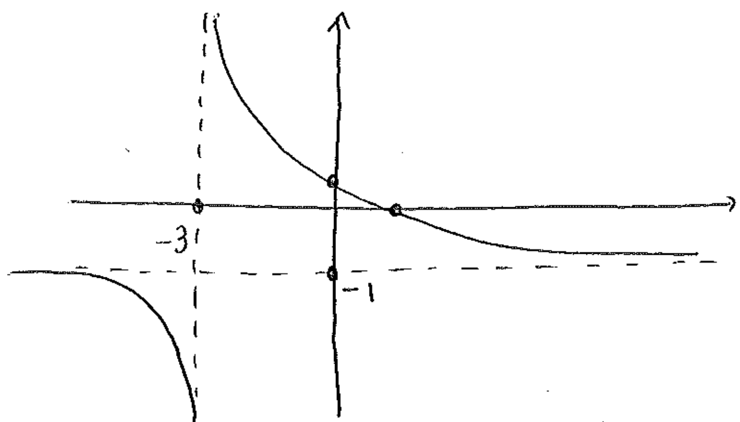
(c) What is the vertical asymptote? $X = -3$

(d) What is the horizontal asymptote? $y = -1$

(e) Draw a signs table for $f(x)$

		-3		1	
$1-x$	$+$		$+$	0	$-$
$x+3$	$-$	0	$+$		$+$
	$-$	0	$+$	0	$-$

(f) Using this information, sketch $f(x)$.



(g) Calculate the inverse of $f(x)$. $y = \dfrac{1-x}{x+3}$

$$(x+3)y = 1-x$$

$$xy+3y = 1-x$$

$$x(y+1) = 1-3y \rightarrow x = \frac{1-3y}{1+y} \rightarrow f^{-1}(x) = \frac{1-3x}{1+x}$$

(h) Verify that $f\left[f^{-1}(x)\right] = x$.

$$\frac{1-\frac{1-3x}{1+x}}{\frac{1-3x}{1+x}+3} = \frac{\frac{1+x}{1+x}-\frac{1-3x}{1+x}}{\frac{1-3x}{1+x}+\frac{3(1+x)}{1+x}} = \frac{\frac{1+x-(1-3x)}{1+x}}{\frac{1-3x+3(1+x)}{1+x}} = \frac{\frac{4x}{1+x}}{\frac{4}{1+x}}$$

$$= \frac{\cancel{4}x}{\cancel{1+x}} \cdot \frac{\cancel{1+x}}{\cancel{4}} = x \checkmark$$

PROBLEM 3: APPLIED PROBLEM [20 POINTS]

Bacteria are grown in a petri dish for an experimental trial. At the beginning of the trial (that is, at time $t = 0$), there are exactly 2 bacteria. Every 5 minutes, each bacteria in the petri dish divides into two.

1. How many bacteria will be present in the dish at time $t = 5$ minutes? 4

2. How many bacteria will be present in the dish at time $t = 10$ minutes? 8

3. How many bacteria will be present in the dish at time $t = 5n$ minutes, where n is an integer ?

$n=1$: $t=5 \to 4 = 2^2$

$n=2$ $t=10 \to 8 = 2^3$

$n=3$ $t=15 \to 16 = 2^4$

$n=4$ $t=20 \to 32 = 2^5$

$\vdots$

n $t=5n \to 2^{n+1}$

$\Rightarrow$ Number of bacteria at time $t=5n$ is 2^{n+1}

4. The number of bacteria in the dish N as a function of time t is (circle the correct answer)

- $N(t) = 2^{t+1}$
- $N(t) = 2^{\frac{t}{5}+1}$
- $N(t) = 2^{\frac{t}{5}}$

$n = \frac{t}{5}$ so $N(n) = 2^{n+1}$ becomes $N(t) = 2^{\frac{t}{5}+1}$

5. Express the $N(t)$ you chose as a natural exponential. $N(t) = 2 \cdot 2^{\frac{t}{5}} = 2 \cdot e^{\frac{t}{5}\ln 2}$ (or $e^{(\frac{t}{5}+1)\ln 2}$)

6. We want to write $N(t)$ as $N(t) = N_0 e^{rt}$ where r is the growth rate of the bacteria.

What is N_0? 2

What is r? $\frac{\ln 2}{5}$

$N_0 e^{rt} = 2e^{\frac{t}{5}\ln 2} \to$ identify $N_0 = 2$, $r = \frac{1}{5}\ln 2$.

7. How long would it take to have 2 million bacteria? Hint: write 1 million as 10^6 and use the fact that $\ln 2 \simeq 0.7$ and $\ln(10) \simeq 2.1$.

$\cancel{2}e^{\frac{t}{5}\ln 2} = \cancel{2}10^6 \Rightarrow e^{\frac{t}{5}\ln 2} = 10^6$

$\ln(e^{\frac{t}{5}\ln 2}) = \ln(10^6) = 6\ln 10 \Rightarrow \frac{t}{5}\ln 2 = 6\ln 10$

$\Rightarrow t = \frac{30\ln 10}{\ln 2} = \frac{30 \times 2.1}{0.7} \simeq 90$ minutes.