Chapter 4

Powers functions. Composition and Inverse of functions.

4.1 Power functions

4.1.1 Definition and examples

Definition:

Examples:

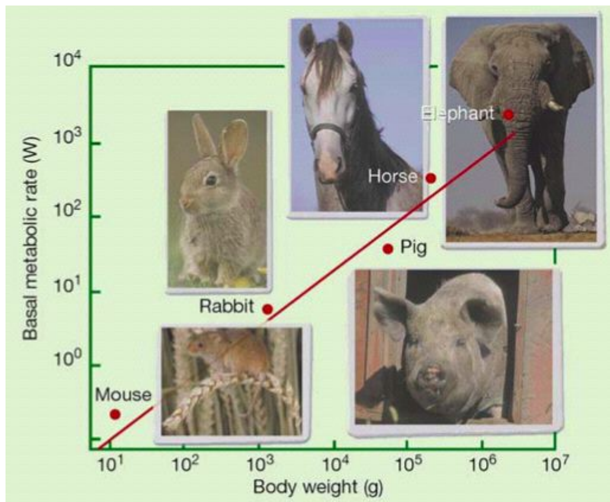
4.1.2 Examples of power functions in Nature

We saw earlier that many geometric problems lead to power-law relationships between variables. However, even in Nature there are many examples of power laws, sometimes with integer exponents, sometimes with non-integer exponents.

- Fundamental forces of Physics follow power-law scalings:
  Example: The force of gravitation

- Allometric laws in Biology: Power functions are frequently found when relating (empirically) two biological variables.

  - EXAMPLE: Kleiber's Laws:
    Body Metabolism Rate = $3.5(\text{Body Weight})^{3/4}$Watts .



  - The optimal cruising speed for a bird/plane as a function of their body mass:
    Speed = $30\text{Mass}^{1/6}$m/s (mass in kg)

4.1.3 Manipulations of power functions

The following rules apply for manipulating power functions:

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4.1.4 Graphs of power functions

The overall shape of the graph of a power function depends on the sign and value of the exponent...

4.2 Composition of functions

So far, we have always just considered functions individually, i.e. we were given a function $f(x)$ to study. As we saw in the very first Chapter, given *two* functions we can add them, subtract them, multiply or divide them and create another function in the process. These are simple *algebraic* manipulations of functions, involving basic algebraic operations ($+$, $-$, $\times$, $\div$)

However, there is another operation on functions that we have so far ignored, which is called *composition of functions*. The idea behind the composition of function comes from the fact that things can depend on each other not only directly, but also through a series of causal links: a quantity A depends on a quantity B, and B depends on C, so indirectly, A also depends on C.

EXAMPLES:

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Since a function can be viewed as dependence relationship (i.e. the dependent variable depends on the independent one), any indirect dependence can also be represented as a function. To do this mathematically, we use an *operator* on functions called *composition*.

EXAMPLES:

- $f(x) = \sin(x)$, $g(x) = 4x - 1$: $f \circ g$:

- $f(x) = \frac{1}{x^2-2}$, $g(x) = x + 1$: $f \circ g$:

- $f(x) = \sqrt{1-x}$, $g(x) = x^2$: $g \circ f$:

In fact, we have already been using functions that are the results of the composition of two other functions all along, without realizing it! For instance,

- $f(x) = x^3$, $g(x) = 4x - 1$: $f \circ g$:

- $f(x) = \frac{1}{x}$, $g(x) = 4x - 1$: $f \circ g$:

Important note: Changing the order of the composition yields an entirely different function!

Example: $f(x) = \sqrt{x}$, $g(x) = x^2 + 1$

- $f \circ g$:

- $g \circ f$:

4.3 Inverse of functions

Textbook section 4.1

4.3.1 Definition and examples

Definition:

As a result

Graphical interpretation:

Examples

- $y = f(x) = 3x + 2$:

- $y = f(x) = x^2$ (for $x \geq 0$):

- $y = f(x) = \sqrt{x-2}$ (for $x \geq 2$):

IMPORTANT NOTES:

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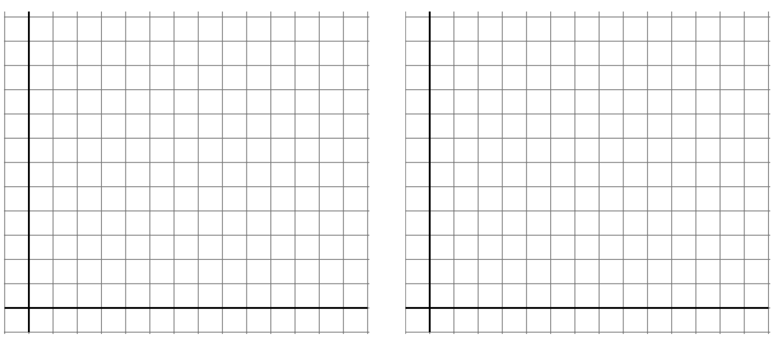
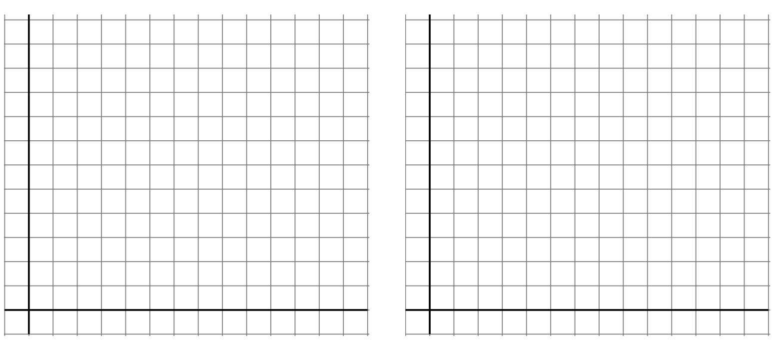
4.3.2 Graph of an inverse function and horizontal line test:

EXAMPLE 1: $y = f(x) = 3x + 2$



EXAMPLE 2: $y = f(x) = x^2$



So from these graphs we notice that:

NOTE: It may happen that through this process, the graph of the inverse does not satisfy the vertical line test: in that case, the inverse is not defined.

HORIZONTAL LINE TEST: To verify that the inverse of a function is unique, we check that the function satisfies the horizontal line test:

When a function $f(x)$ does not satisfy the horizontal line test, we can often choose a smaller domain for which the inverse *is* unique.

EXAMPLE: for the function $f(x) = x^2$, we saw earlier that the inverse of $f(x) = x^2$ is defined provided we select only the interval for which $x \geq 0$. In this interval, the function $f(x)$ does satisfy the horizontal line test.

4.3.3 Inverse of simple rational functions (a worked example)

What is the inverse of the function $f(x) = \frac{3x+1}{x-2}$?

4.3.4 Inverses of power functions

RULE:

PROOF 1:

PROOF 2: Remember that to show that two functions (f and g for example) are inverse of one another, you simply evaluate $f \circ g$:

APLICATIONS:

- What is the inverse of $f(x) = x^3$?
- What is the inverse of $f(x) = x^{-1/2}$?
- What is the inverse of $f(x) = x^{-\pi}$?
- Solve for x: $x^{2\pi} - 5x^{\pi} + 6 = 0$

- What is the typical mass of a mammal whose metabolic rate is 100 Watts?