3.2   Inequalities involving polynomials and rational functions

Textbook section 3.6

Now that we know a fairly extensive range of basic functions, we have the tools to study inequalities in more detail.

3.2.1   What does "solving" inequalities mean?

DEFINITION:

EXAMPLES:

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There are two basic methods for solving inequalities. One is inexact (or numerical), and relies on plotting the graphs of the functions on both sides and finding out when one is greater or smaller than the other. The other method is exact (or analytical) and involves manipulating the inequalities and then making use of signs tables. The first one always works, but is not always practical (especially if we don't have a graphing device at hand); the second is usually limited to problems that involves rational or polynomial functions. Let's briefly look at the first one, then study the second one in more detail.

3.2.2   "Graphical" method for solving inequalities

A very powerful method for solving inequalities consists in plotting the graphs of the functions on both sides of the inequality, and seeing when one of the functions lies above the other.

Given two functions $f(x)$ and $g(x)$:

EXAMPLE: Given the graphs of $f(x) = x + 2$ and $g(x) = \frac{1}{x-1}$, determine when $f(x) > g(x)$.
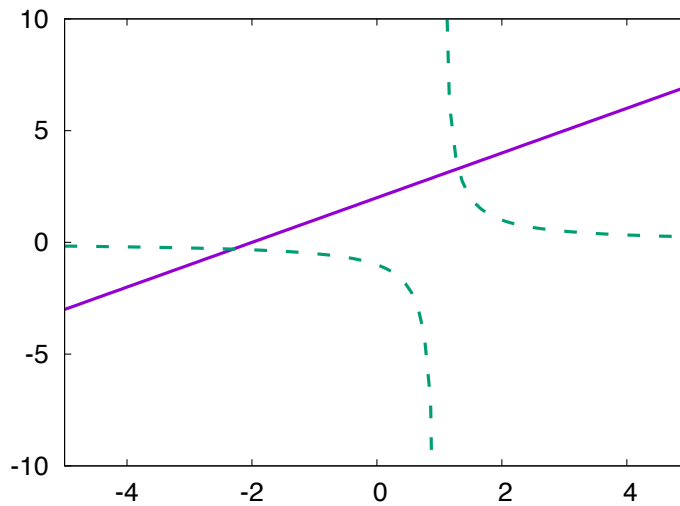
In some cases (as in the example above, but certainly not always), once we have *roughly* identified the right intervals that form the solution of the *inequality*, we can sometimes get a proper solution by solving the equation $f(x) = g(x)$, since the *equality* defines the points where the two graphs intersect.

EXAMPLE: Solve the inequality $x^2 + 2x + 1 \geq 1$

EXAMPLE: Solve the inequality $|x + 1| \geq -2x$

3.2.3 "Direct" method for solving inequalities

The direct method for solving inequalities relies on manipulating the inequality to put in a form that just looks for the sign of a function (i.e. positive or negative), and then to use signs tables to determine in which intervals the function is indeed positive or negative). For instance,

while

Then the problem boils down to factoring $f(x) - g(x)$. Note that this second step is not always possible.

Let's first look again at the simple example we did earlier: Solve the inequality $x^2 + 2x + 1 \geq 1$.

This method can be used to solve almost all polynomial and rational inequalities, provided the resulting function $f(x)$ can be factored. Here are some more examples.

EXAMPLE 1: Solve the inequality $\frac{3x-2}{x+1} \leq 1$.

EXAMPLE 2: Solve the inequality $x^3 - 2x^2 > x - 2$

3.2.4   A tricky trap !

The first example in the previous section holds a nasty trap, that's easy to fall into. Naively, one might want to do the problem a different way:

But that gives a different answer! As it turns out, this one is incorrect, and the only correct way of doing this problem is to do it the way we did in the previous section. The reason this way doesn't work is quite tricky, and is because *we must be extremely careful when multiplying inequalities by expressions of unknown sign.* Here, we multiplied it by $x + 1$, but $x + 1$ can either be negative (if $x \leq -1$) or positive (if $x \geq -1$). While it's ok to multiply an inequality by a positive number, when we multiply it by a negative number we must also change its direction. And since we don't know a priori what sign $x + 1$ has, we cannot multiply the inequality by $x + 1$ without risking making a mistake. Meanwhile, the way we did it in the first place works because we are only subtracting $g(x)$ on both sides – and subtraction is an operation that can always be done without affecting the direction of the inequality.

3.2.5   Rules of manipulation of inequalities

Manipulating inequalities is often something students struggle with. The problem is that one would *like* to be able to manipulate inequalities the same way we manipulate equations, but it's not always straight-forward: as we saw earlier there are some operations on inequalities that preserve the direction of the inequality (which is fine) but other operations that *change* the direction of the inequality (which makes

things tricky). The question is: *how do we know when to preserve and when to change the direction of the inequality?*

THE RULES OF MANIPULATIONS OF INEQUALITIES. The standard rules are the following:
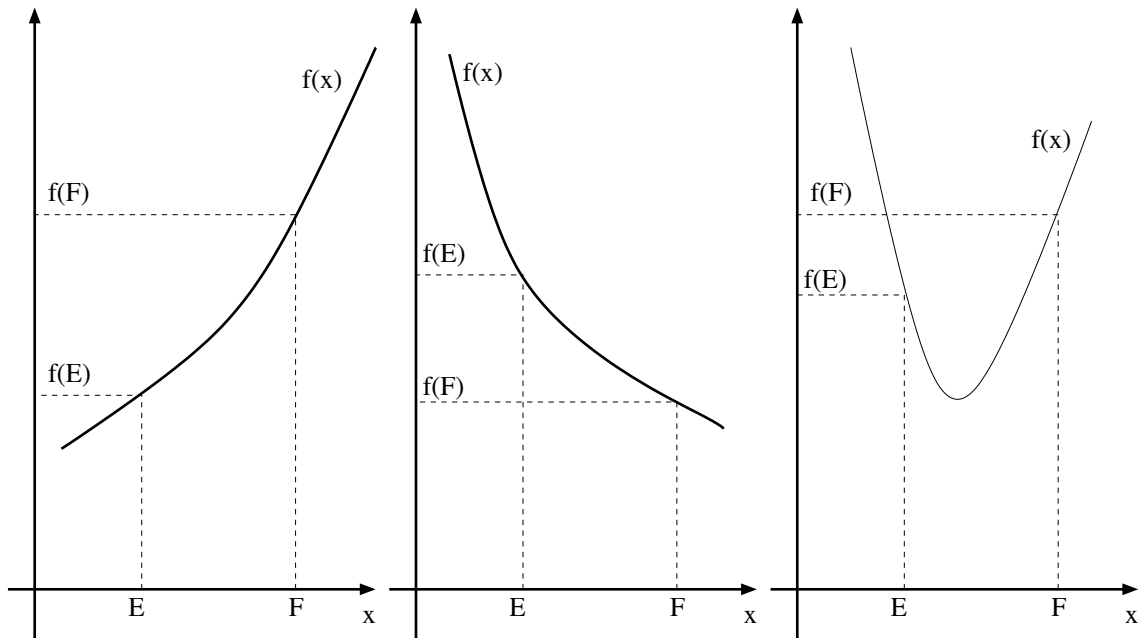
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EXAMPLES:

WHERE DO THESE RULES COME FROM, AND CAN WE DO OTHER OPERATIONS ON INEQUALITIES?

Imagine you have an inequality $E \leq F$ where E and F can be any expressions (numbers, variables, functions, etc..). Then, if you want to transform the inequality with a transformation rule f, note that

- $E \leq F \Rightarrow f(E) \leq f(F)$ if f is an increasing function in $[E, F]$

- $E \leq F \Rightarrow f(E) \geq f(F)$ if f is a decreasing function in $[E, F]$

- We don't know what happens if f has a slope that changes in $[E, F]$.

Examples:

- Suppose you want to add or subtract a constant (say, 2, or 3, or π) to both sides of an inequality.

- Suppose you want to multiply by a positive constant (say 2, 4, etc..) on both sides of an inequality.

- Suppose you want to multiply by a *negative* constant (say -2, -0.1, etc..) on both sides of an inequality.

This can now be generalized to *any operations!*

MORE EXAMPLES: Suppose you have the inequality

$$x \leq y$$

and both x and y are greater than 0. Using the relevant graph, assess whether the following statements are true or false?

- $x^2 \leq y^2$

- $x^3 \leq y^3$

- $\frac{1}{x} \leq \frac{1}{y}$

- $\sqrt{x} \leq \sqrt{y}$