2.3 Higher-order polynomials

Textbook sections 3.1-3.2

2.3.1 Definition and examples

DEFINITION: A polynomial function in expanded form is any function of the kind $f(x) = a_0 + a_1 x + a_2 x^2 + \dots a_n x^n$ where $n \geq 0$ and $a_0, a_1, a_2 \dots a_n$ are real numbers

VOCABULARY:

- n is called the order of the polynomial
- $a_n x^n$ is called the leading term

EXAMPLES:

$f(x) = 4x^2 - 3x + 1$: order 2 (also called quadratic)
$a_2 = 4$ $a_1 = -3$ $a_0 = 1$ $n = 2$

$f(x) = x^3 - 2x$: order 3 (cubic) $a_3 = 1$, $a_2 = 0$
$a_1 = -2$ $a_0 = 0$

$f(x) = -x^4 + 3x^3 - 2x^2 + x + \pi$ order 4 (quartic) ...

In order to understand the behavior of these polynomials, let's start with studying the behavior of individual terms: functions which are simple powers of x.

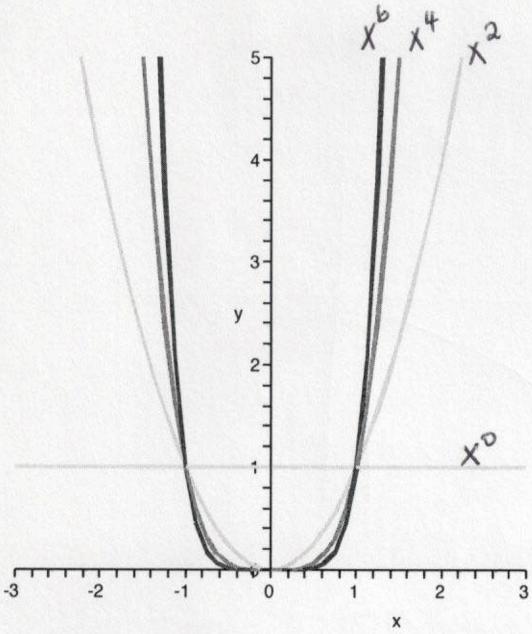
2.3.2 Power functions of the kind $f(x) = ax^n$ with n a natural number

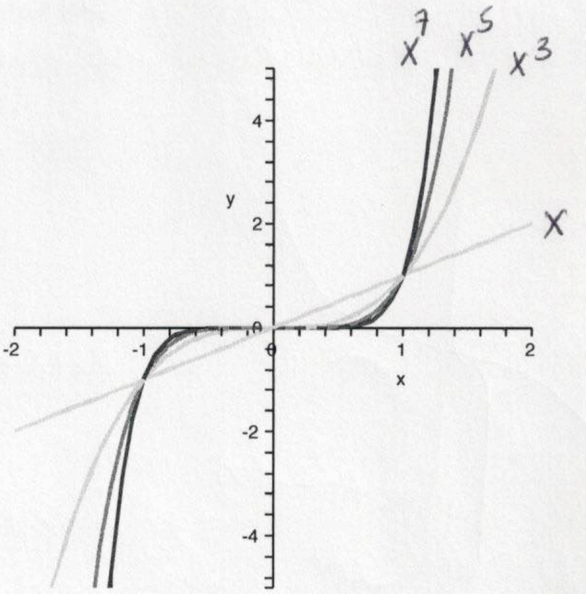
REAL-LIFE ax^n FUNCTIONS: Power functions with integer powers arise naturally in geometrical problems.

- The surface area of a cube as a function of side length: $S(\ell) = 6\ell^2$
- The circumference of a circle as a function of radius: $C(r) = 2\pi r$
- The area of a circle as a function of radius: $A(r) = \pi r^2$
- The volume of a sphere as a function of radius: $V(r) = \frac{4}{3}\pi r^3$

They also occur in allometric laws in nature:

- Metabolic rate: $R(m) = \text{Const} \cdot m^{3/4}$ (m=mass)
- Heart rate: $H(m) = \text{const} \cdot m^{1/4}$





GRAPHS OF POWER FUNCTIONS: The shape of the graphs of functions of the kind $f(x) = x^n$ depends on whether n is an even or an odd number (see above).

NOTE:

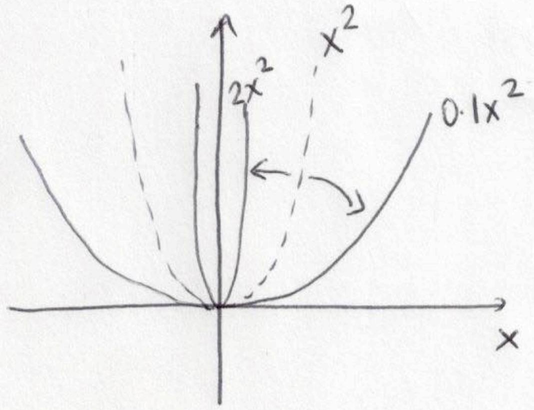
- All functions $f(x) = x^n$ go through $(1,1)$ because $f(1) = 1^n = 1$

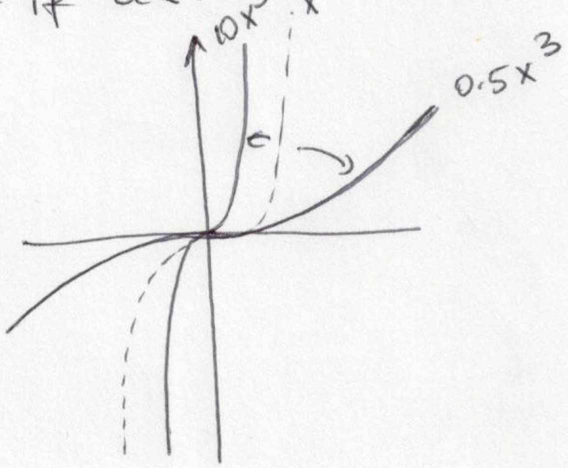
because $f(-x) = (-x)^n = (-1)^n x^n = \begin{cases} x^n & \text{if } n \text{ even} \\ -x^n & \text{if } n \text{ odd} \end{cases}$

- Functions with n even are symmetric about y-axis → they are <u>even</u> functions
- Functions with n odd are point-symmetric about $(0,0)$ → they are <u>odd</u> functions.

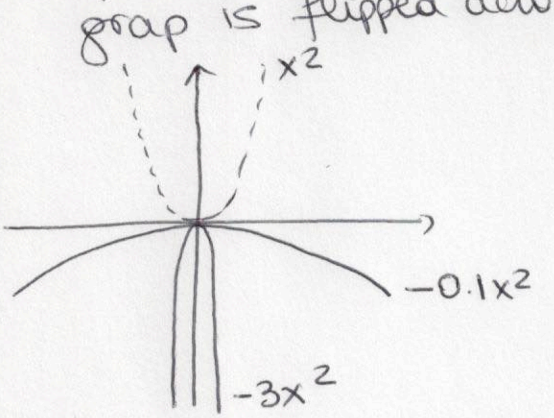
When the power is multiplied by a number a, note that

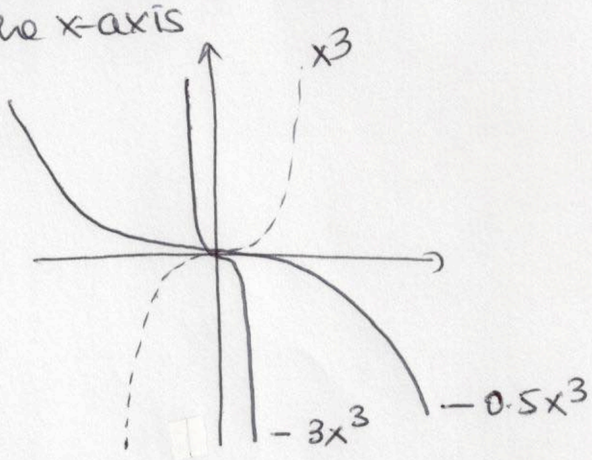
- if $a>0$ the overall shape is similar but generally
  - steeper if $a>1$
  - shallower if $a<1$





- if $a<0$ the same is true but in addition, the graph is flipped across the x-axis





2.3.3 Approximations of polynomials for very large values of $|x|$

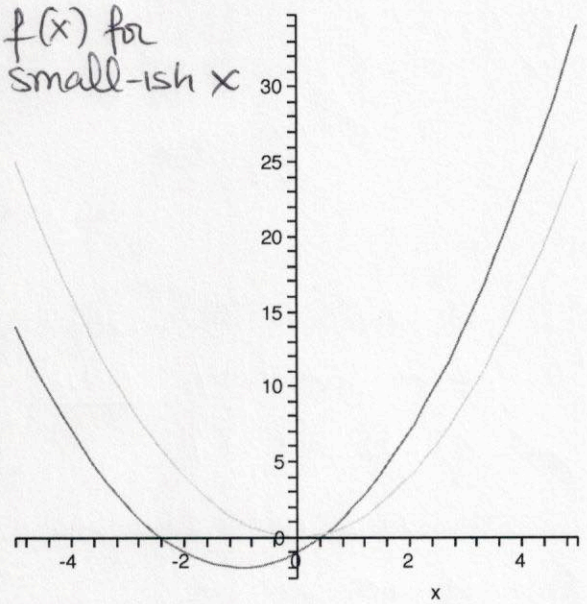
The overall shape of the graph of a polynomial function, when x is either very large and positive (x going to $+\infty$) or very large and negative (x going to $-\infty$), resembles that of the corresponding power function of the leading order term.

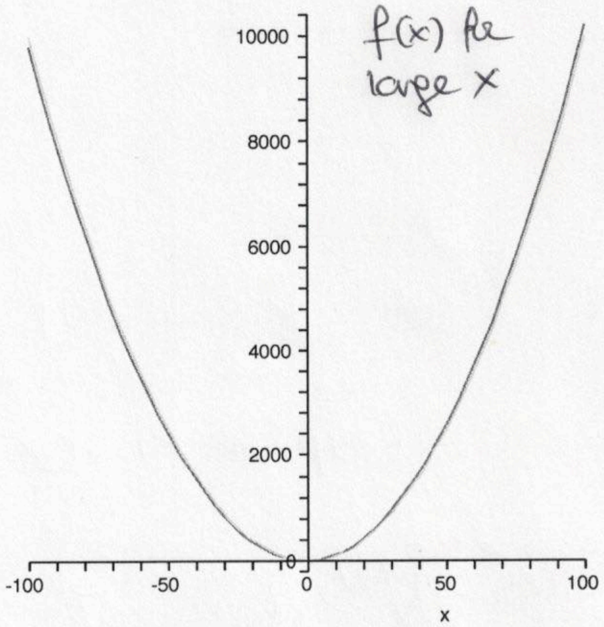
In other words, when $|x| \to +\infty$,

$$f(x) = a_0 + a_1 x + a_2 x^2 + \dots a_n x^n \simeq a_n x^n$$

and the graph of $f(x)$ is close to that of $a_n x^n$ (for large x)

EXAMPLE 1: $f(x) = x^2 + 2x - 1$

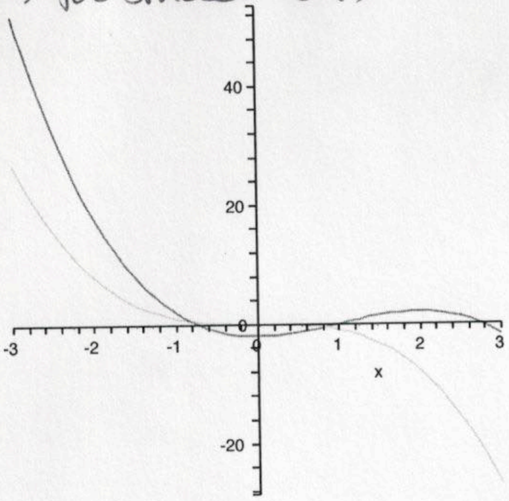




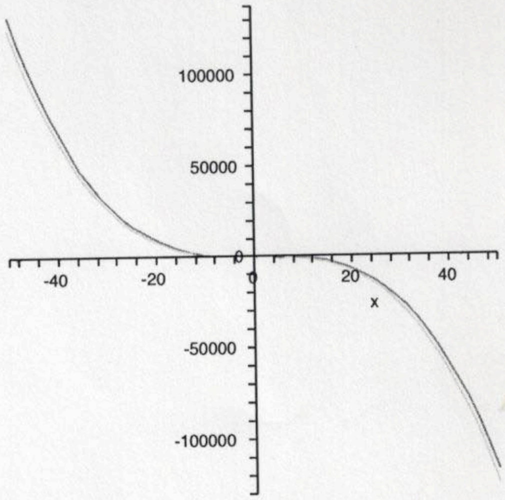
$f(x) = x^2 + 2x - 1$
$\simeq x^2$ for large x $\longrightarrow$ the graph looks like that of x^2 for large x

EXAMPLE 2: $f(x) = -x^3 + 3x^2 - 2$

f(x) for small-ish x



f(x) for large x



$f(x) = -x^3 + 3x^2 - 2 \simeq -x^3$ → the graph looks like that of $-x^3$ for large x

CONCLUSION:

Knowing the graph of <u>power functions</u> tells you about the behavior of polynomial functions for large $|x|$.

2.3.4 Factored polynomials

Polynomial functions can also be expressed in factored form.

FORMAL DEFINITION OF FACTORED FORM:

A polynomial of order n (ie $f(x) = a_0 + a_1 x + \cdots a_n x^n$) is <u>fully</u> factored when it is written as

$$f(x) = a_n (x - x_1)(x - x_2) \cdots (x - x_m)\, q(x)$$

where
- $m \leq n$
- the $x_1, x_2 \ldots x_m$ are roots of $f(x)$ (note that they don't have to be different)
- $q(x)$ is a polynomial whose leading term is x^{n-m}, and it does <u>not</u> have any roots (ie $q(x) \neq 0$ for all x)

It is not always easy to determine whether a polynomial is fully factored, or can be factored further. Sometimes, the polynomial is already obviously fully factored. Sometimes, it is partially factored, and one must decide if the remaining part can be factored further or not. Sometimes the polynomial is fully expanded, and one must start factoring it from scratch.

EXAMPLES:

- $f(x) = -(2+x)(x+3)^3$ → already fully factored.
  $a_4 = -1$, $x_1 = -2$ $x_2 = -3$ $x_3 = -3$ $q = 1$

- $f(x) = (x-1)(2-x^2)$ → not fully factored because we can write $2 - x^2 = (\sqrt{2} - x)(\sqrt{2} + x) = -(x-\sqrt{2})(x+\sqrt{2})$ so

  $f(x) = -(x-1)(x-\sqrt{2})(x+\sqrt{2})$ so
  $a_3 = -1$ $x_1 = +1$ $x_2 = \sqrt{2}$ $x_3 = -\sqrt{2}$ $q = 1$

- $f(x) = -2x(x^2 - 2x + 1)(x+3)$ → not fully factored because $x^2 - 2x + 1 = (x-1)^2 = (x-1)(x-1)$ so

  $f(x) = -2(x-0)(x-1)(x-1)(x+3)$ so
  $a_4 = -2$ $x_1 = 0$ $x_2 = 1$ $x_3 = 1$ $x_4 = -3$

- $f(x) = x^3 + 2x^2 + 4x$ → not fully factored:

  $f(x) = x(x^2 + 2x + 4)$ ⇒ is $x^2 + 2x + 4$ factorable?
  $D = (2)^2 - 4(1)(4) = 4 - 16 = -12$ → no roots so
  $q(x) = x^2 + 2x + 4$ $x_1 = 0$ $a_3 = 1$.
  $f(x) = (x-0)(x^2 + 2x + 4)$

In the examples presented above, it is still reasonably easy to determine whether a polynomial is fully factored or not, either by finding a common factor, or by recognizing one of the standard patterns. However, there are many case in which it is not so easy. There is one more factoring technique that can be used and works sometimes (not always) for more complicated polynomials, and is called the *grouping technique*. The idea behind grouping is that finding a common factor between two pairs of terms is easier than finding a common factor in the whole expression. Then, sometimes, each pair once factored contains a term which is common to all pairs, and can be used as the next common factor.

EXAMPLES:

- $x^3 + 2x^2 - x - 2$ $= x^2(x+2) - (x+2)$
  $= (x+2)(x^2-1) = (x+2)(x-1)(x+1)$
  → now fully factored

- $2x^5 - 3x^4 + 6x^2 - 9x$

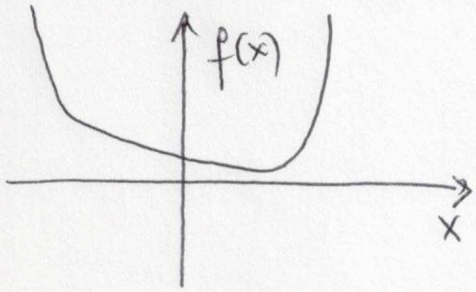
$= x^4(2x-3) + 3x(2x-3)$

$= (2x-3)(x^4+3x) = (2x-3)x(x^3+3)$

Remember, however, that grouping rarely works – only special kinds of polynomials can be factored that way. In fact, for any polynomial of degree 5 or more, there is no simple rule one can apply to factor it, or to determine what its roots are. The only way to do it is to graph it, and see whether it crosses the $x-$axis or not, and how it does it.

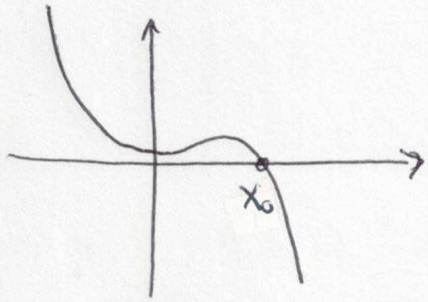
EXAMPLES: Determine graphically whether the following polynomials can be factored, and if they can, what form the factored form would take.

- $f(x) = x^6 - 2x + 3$ Use Wolfram α:



$\Rightarrow$ $f(x)$ has no roots, it's fully factored as is.

- $f(x) = -x^5 + 4x^3 - x + 1$



Looks like $f(x)$ has only one root; by zooming in, it is close to 3.988

$\to f(x) = -(x-x_0)\,q(x)$ is the fully factored form.

Finally, note that there is a very important theorem about the number of roots of a polynomial:

A polynomial ~~of~~ order n has at most n real roots. To see this note that, in the fully factored form, we have (if $q(x)=1$)

$f(x) = a_n \underbrace{(x-x_1)(x-x_2)\cdots(x-x_n)}_{n \text{ roots at most}}$ (assuming they are all different)

2.3.5 Signs tables

Signs Tables are an excellent tool to determine the *sign* of any polynomial function, quadratic or higher-order. Knowing the sign of a function is often a very useful tool for graphing, and for finding out the domain of definition of a function.

IMPORTANT NOTE: Signs tables can only be used if the function is already broken down into its factors.

HOW TO DRAW A SIGNS TABLE:

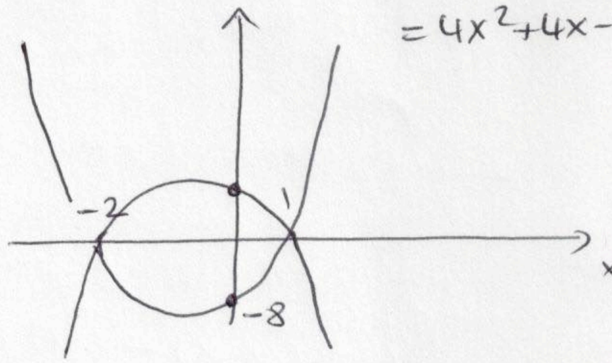
- Draw the table
- Write **all** the factors vertically on the left
- Write **all** the roots horizontally on the top (in the correct order)
- Draw vertical lines below each root
- Determine and write the sign of each factor; write zeros where appropriate.
- Multiply the signs in each interval to determine the sign of the function.

EXAMPLES OF USE OF SIGNS TABLE WITH QUADRATIC FUNCTIONS:

EXAMPLE 1: Draw a signs table and sketch the function $f(x) = 4(x-1)(x+2)$. $= 4(x^2 + x - 2)$

$= 4x^2 + 4x - 8$

		-2		1	
4	$+$		$+$		$+$
$x-1$	$-$		$-$	0	$+$
$x+2$	$-$	0	$+$		$+$
	$+$	0	$-$	0	$+$



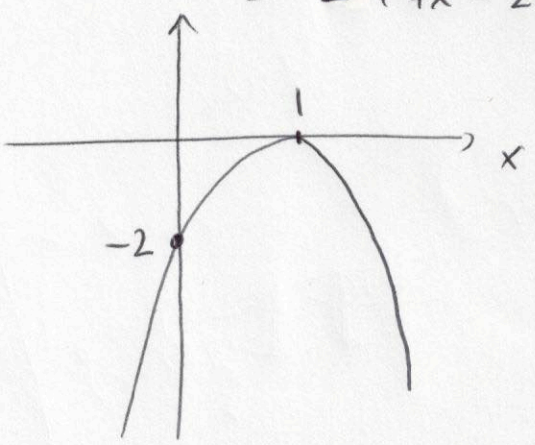
EXAMPLE 2: Draw a signs table and sketch the function $f(x) = -2(1-x)^2$. $= -2(1 - 2x + x^2)$

$= -2 + 4x - 2x^2$

		1	
-2	$-$		$-$
$1-x$	$+$	0	$-$
$1-x$	$+$	0	$-$
	$-$	0	$-$

alternatively

		1	
-2	$-$		$-$
$(1-x)^2$	$+$	0	$+$
	$-$	0	$-$



EXAMPLE 3: What is the domain of definition of $f(x) = \sqrt{2x^2 + 2x - 1}$?

Here we are looking for the sign of $2x^2+2x-1$

→ first need to factor this quadratic

$D = (2)^2 - 4(2)(-1) = 4 + 8 = 12$ so there are 2 roots,

$$x_{1,2} = \frac{-2 \pm \sqrt{12}}{4} = -\frac{1 \pm \sqrt{3}}{2} = \begin{cases} -1.366 = x_1 \\ 0.366 = x_2 \end{cases}$$

So $2x^2+2x-1 = 2(x-x_1)(x-x_2)$

		x_1		x_2	
2	+		+		+
$x-x_1$	−	0	+		+
$x-x_2$	−		−	0	+
$2x^2+2x-1$	+		−		+

so $2x^2+2x-1$ is positive when $x \leq x_1$ or $x \geq x_2$

$\Rightarrow \mathcal{D} = \left(-\infty, \frac{-1-\sqrt{3}}{2}\right] \cup \left[\frac{-1+\sqrt{3}}{2}, +\infty\right)$

HIGHER-ORDER POLYNOMIAL FUNCTIONS:

To create a signs table for a fully factored higher-order polynomial, simply follow the same method as for quadratic functions. The information obtained can be very useful for a number applications:

EXAMPLES:

- In which interval(s) is the function $f(x) = -(2+x)(x+3)^3$ positive?

→ already fully factored

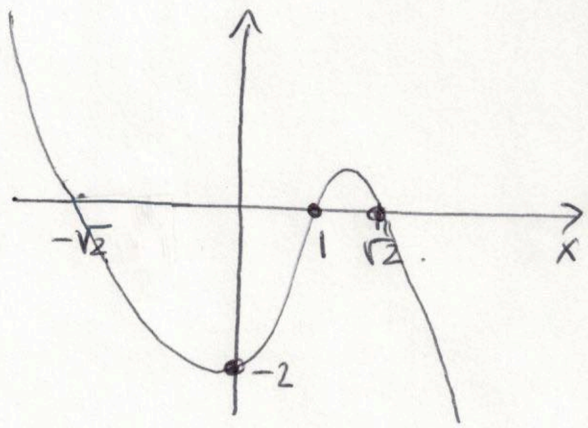
		-3		-2	
$2+x$	−		−	0	+
$x+3$	−	0	+		+
$x+3$	−	0	+		+
$x+3$	−	0	+		+
-1	−		−		−
	−	0	+	0	−

⇒ it is positive when

$-3 \leq x \leq -2$

- Sketch the graph of the function $f(x) = (x-1)(2-x^2)$. $= (x-1)(\sqrt{2}-x)(\sqrt{2}+x)$

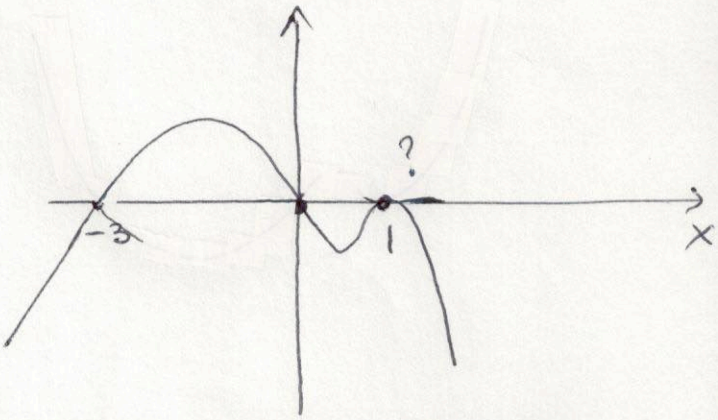
		$-\sqrt{2}$		1		$\sqrt{2}$	
$x-1$	−		−	0	+		+
$\sqrt{2}-x$	+		+		+	0	−
$x+\sqrt{2}$	−	0	+		+		+
	+	0	−	0	+	0	−



$f(x) = 2x - x^3 - 2 + x^2$
$= -x^3 + x^2 + 2x - 2$

- Sketch $f(x) = -2x(x^2 - 2x + 1)(x + 3)$ $= -2x(x-1)^2(x+3)$

		-3		0		1	
$-2x$	+		+	0	−		−
$(x-1)^2$	+		+		+	0	+
$(x+3)$	−	0	+		+		+
	−	0	+	0	−	0	+



- Find the domain of definition of $f(x) = \sqrt{x^3 + 2x^2 + 4x}$. $= \sqrt{x(x^2+2x+4)}$

x^2+2x+4 has no roots → it always has the same sign. At $x=0$ it is positive so it is always positive

		0	
x	−	0	+
x^2+2x+4	+		+
	−	0	+

$\longrightarrow \mathcal{D} = [0, +\infty)$

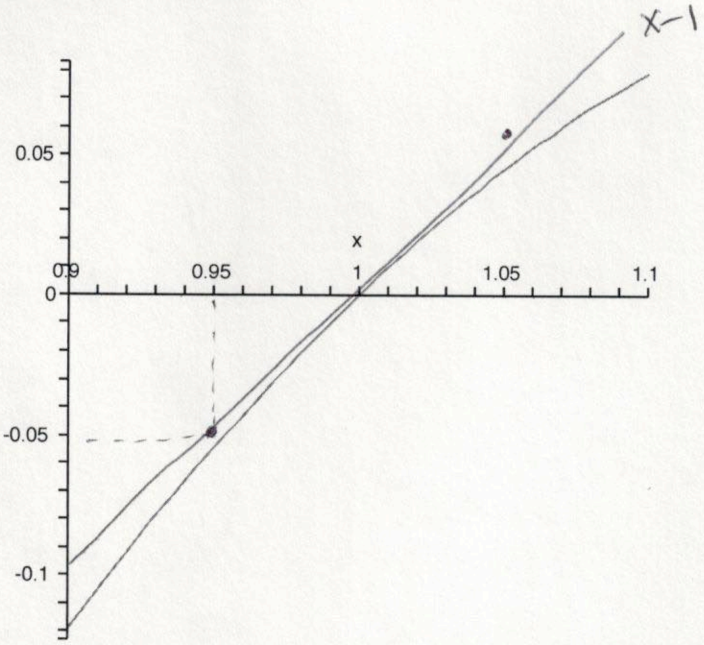
Behavior near a root

While the signs table typically gives you most of the information you need about the function, we saw that in some cases the situation is not so clear-cut. In these cases, it is also useful to study the behavior of the function *in the vicinity* of a root, to double-check the signs table and sometimes to find out more about the function.

EXAMPLE 1: Consider the function $f(x) = (x - 1)(2 - x^2)$. Near $x = 1$, of course, $f(x)$ is close to 0 (because the $x - 1$ term becomes very small). But what does it look like?
Since x is close to 1, let's see what happens if we plug $x = 1$ into *all the factors except* $(x - 1)$. Then we get

$$(x-1)(2-x^2) \simeq (x-1)(2-1) \cong (x-1)$$

When plotted on the same plot, the two functions are indeed very close to each other for x near 1.



So the function $f(x) = (x-1)(2-x^2)$ indeed looks like $x-1$ near $x=1$

NOTE: This can also help us check the signs table: the table says that $f(x)$ goes from negative to positive as x goes through 1. The line $y = x - 1$ also goes from negative to positive as x goes through 1.

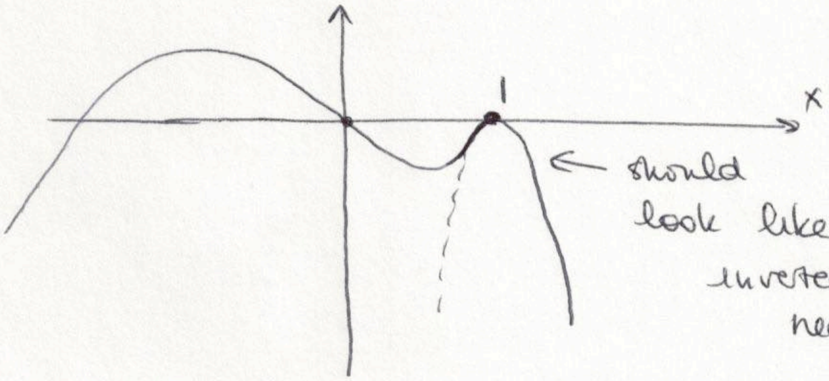
EXAMPLE 2: Consider the function $f(x) = -2x(x^2 - 2x + 1)(x + 3)$. What does it look like near $x = 1$?

$$f(x) = -2x(x-1)^2(x+3)$$

→ we plug $x=1$ everywhere <u>except</u> in the term that has $x-1$:

$$f(x) \simeq -2(1)(x-1)^2(1+3)$$
$$\simeq -8(x-1)^2$$

so near $x=1$, $f(x)$ looks like $-8(x-1)^2$



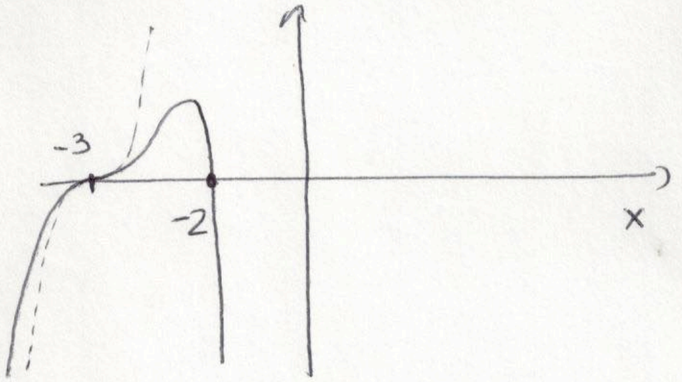
← should look like an inverted parabola $y=-8(x-1)^2$ near $x=1$.

EXAMPLE 3: Consider the function $f(x) = -(2+x)(x+3)^3$. What does it look like near $x = -3$?

Plug $x=-3$ in $f(x)$ except in the term with $(x+3)$

$\to f(x) \simeq -(2-3)(x+3)^3 = (x+3)^3$

$\to f(x)$ looks like $(x+3)^3$ near $x=-3$



Using the signs table from earlier, we get this sketch.

We can now check with a graphing device that our guesses are indeed correct: