2.2.7 Roots of quadratics

As we saw in the last lecture, some parabolas cross the $x-$axis, and some do not. Graphically, we know that there are 3 possible cases: a parabola can never crosses the $x-$axis, it can just grazes the $x-$axis at one point, or it crosses it at two different points:

Since a parabola is the graph of a quadratic function, this means that some quadratic functions $f(x) = ax^2 + bx + c$ have 2 roots, some have one root, and some do not. In other words,

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Whether a quadratic has one or two roots is is much more obvious from its factored form than from its expanded form.

Examples

- $f(x) = 3(x-1)(x+2)$

- $f(x) = \frac{1}{2}(x-4)^2$

More generally:

In fact, there is a strict equivalence relationship between the two statements "$f(x)$ can be factored" and "$f(x)$ has roots": the first implies the second, and conversedly, the second implies the first. This is often written mathematically as:

The interesting thing about equivalence statements in logic is that if you have one, then you also have equivalence of the opposites:

For the case of factored forms and roots, the "opposite statements" then give

Examples

- $f(x) = x^2 + 4$

- $f(x) = x^2 + 2x + 4$

2.2.8 Factoring quadratics

Based on what we just saw, it would be nice to have simple tricks to tell us when a quadratic has roots or not, and what they are. This would then automatically tell us when a quadratic can be factored, and what the factored form is.

As we saw in the prevoius section, there are a few types of quadratics that can very easily be factored:

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EXAMPLES

- $f(x) = x^2 - 2$

- $f(x) = 2x^2 - 3$

- $f(x) = x^2 + 6x + 9$

- $f(x) = 2x^2 + 4\sqrt{5}x + 10$

- $f(x) = -x^2 + 10x - 25$

On the other hand, not every quadratic is in one of these three "ideal forms". What can we do if it isn't? As it turns out, another nice trick exists in that case, and is called "The quadratic formula".

THE QUADRATIC FORMULA. Given the quadratic $ax^2 + bx + c$,

- Calculate the discriminant $D = b^2 - 4ac$
- If $D < 0$ there are no solutions to the equation $ax^2 + bx + c = 0$, and the quadratic cannot be factored.
- If $D = 0$ there is one solution to the equation $ax^2 + bx + c = 0$, $x = -\frac{b}{2a}$ and the quadratic can be factored as

- If $D > 0$ there are two solutions to the equation $ax^2 + bx + c = 0$, which are

  and the quadratic can be factored as

EXAMPLES:

- What are the solutions (if any) to the equation $f(x) = 2x^2 - 3x + 1 = 0$? What is the factored form of f?

- What are the solutions (if any) to the equation $f(x) = x^2 + x - 6 = 0$? What is the factored form of f?

- What are the solutions (if any) to the equation $f(x) = -2x^2 - 8x - 8 = 0$? What is the factored form of f?

- What are the solutions (if any) to the equation $f(x) = -x^2 + x - 6 = 0$? What is the factored form of f?

NOTE: In a few particular cases, this method can also help solve higher-order equations that can be reduced to a quadratic, as in these examples:

- What are the solutions (if any) to the equation $f(x) = x^6 - 3x^3 - 9 = 0$?

- What are the solutions (if any) to the equation $f(x) = x^4 - 2x^2 - 3 = 0$?

WHERE DOES THE QUADRATIC FORMULA COME FROM?

Understanding where the quadratic formula comes from is important because it makes use of neat mathematical tricks, and also gives you a way to "rediscover it" should you ever forget it.

STEP-BY-STEP: From $ax^2 + bx + c$

- Factor a:

- Complete the square in bracket

- Evaluate the discriminant $D = b^2 - 4ac$

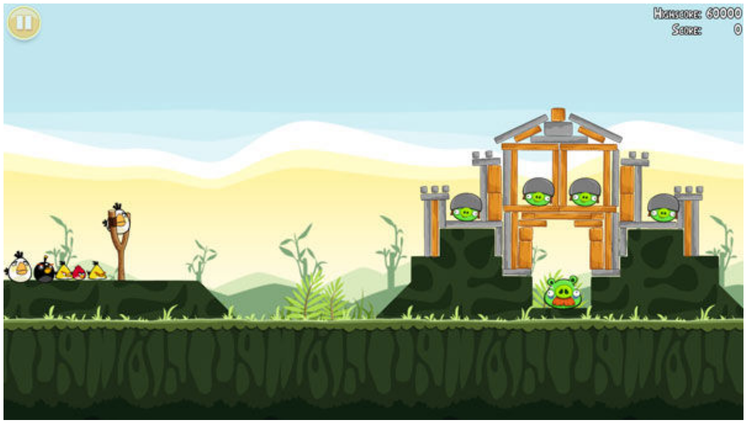
- If $D < 0$ then

- If $D = 0$

- If $D > 0$

2.2.9 A fun application for quadratics

Quadratics were first studied seriously when it was realized that they universally describe the trajectory of thrown objects (c.f. Isaac Newton's work). Let's consider the following scenario...



If the white bird is thrown from 1m off the ground, at a velocity of 1m/s, and at a 45 degree angle from the horizontal, how far ahead will it land?

To answer this question, it may help to know that the trajectory of an object thrown at a velocity v_0 (in m/s), and angle α from the horizontal, and from a height h_0, is given by