2.2 Quadratic functions

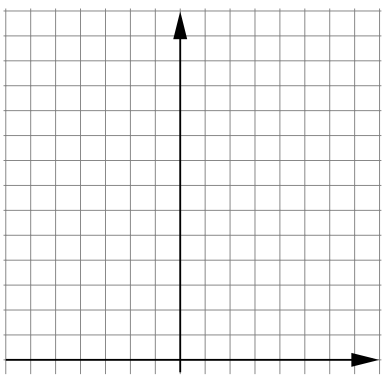
Textbook Sections 2.3, 2.4 and 2.6

2.2.1 Definition and basic $f(x) = x^2$ function

The general expression for a quadratic function is

Some quadratic functions (but not all of them!) can be factored, so that:

The simplest example of a quadratic function is the function $f(x) = x^2$:



In fact, the graph of all quadratic functions is a *parabola*. The exact shape and position of the parabola depends on the coefficients of the quadratic. Different cases can arise:

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In the next few sections, we will get a better intuition for the relationship between the graph of a parabola and the mathematical expression of the corresponding function.

2.2.2 Behavior as $x \to \pm\infty$

Whether a parabola opens "up" or "down" can very easily be determined simply by inspection of the quadratic term ax^2 in the function.

Let's consider two examples of quadratic functions:

- $f(x) = 3x^2 - 2x - 1$
- $g(x) = -2x^2 + x + 1$

and complete the following tables:

x	$3x^2$	$-2x$	-1	$f(x) = 3x^2 - 2x - 1$
-1000				
-100				
-10				
-1				
0				
1				
10				
100				
1000				

x	$-2x^2$	x	1	$g(x) = -2x^2 + x + 1$
-1000				
-100				
-10				
-1				
0				
1				
10				
100				
1000				

We notice that:

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This is in fact true of all quadratics!

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2.2.3 Behavior as $x \to 0$

What the parabola looks like near the $y-$ axis (i.e. when x is close to 0) can *also* very easily be determined simply by inspection of the quadratic function, but this time, of the $bx + c$ bit.

Let's complete the following tables for $f(x)$ and $g(x)$ again, but this time, for small x.

x	$3x^2$	$-2x$	-1	$f(x) = 3x^2 - 2x - 1$	$-2x - 1$
-1					
-0.1					
-0.01					
0					
0.01					
0.1					
1					

x	$-2x^2$	x	1	$g(x) = -2x^2 + x + 1$	$x + 1$
-1					
-0.1					
-0.01					
0					
0.01					
0.1					
1					

We notice that:

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Again, this is true for every quadratic!

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2.2.4 Vertex of a parabola

DEFINITION:

As we shall see shortly, there are many examples of application where we are interested in finding out what the coordinates of the vertex are. As it turns out, it is quite easy to find them, using a method called *completing the square*. To see how it works, let us first note that, in some cases, it's actually very easy to find the vertex without any mathematical manipulations at all, simply by remembering the basic rules of graphical transformations

- $f(x) = x^2$:

- $f(x) = x^2 - 1$

- $f(x) = -3x^2 + 4$

- $f(x) = (x - 2)^2$

- $f(x) = 1 + (x + 3)^2$

- $f(x) = -5(x - 3)^2 - 1$

In all of these examples we see that if the quadratic function $f(x)$ is in the *vertex* form

then:

2.2.5 Completing the square

"Completing the square" in a quadratic expression basically means putting it in vertex form, ie., rewriting $f(x) = ax^2 + bx + c$ as $f(x) = a(x - x_v)^2 + y_v$.

Note how the a is the same in both expressions – do you see why?

To complete the square in practice involves:

- matching $ax^2 + bx$ to the first two terms of the expanded vertex form, to get x_v
- then figuring out what y_v has to be to match c

EXAMPLE 1: $f(x) = x^2 - 2x + 3$.

EXAMPLE 2: $f(x) = 3x^2 - 3x - \frac{1}{4}$.

EXAMPLE 3: $f(x) = -\frac{1}{2}x^2 - 3x + 1$.

Another method for completing the square also exists, that does not require memorizing the two formulas for the $x-$ and $y-$positions of the vertex. They do, on the other hand, require that you recognize some of the standard formulas for expanding quadratics, as in

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So, if we have the beginning of the expressions on the right, we can create the "perfect squares" on the left!

Example 1: $f(x) = x^2 + 2x$.

Example 2: $f(x) = 2x^2 - 2x$.

Example 3: $f(x) = x^2 - 2x + 3$.

Example 4: $f(x) = 3x^2 - 3x - \frac{1}{4}$.

Example 5: $f(x) = -\frac{1}{2}x^2 - 3x + 1$.

2.2.6 Application: optimization problems

Exercise 9 page 171. Also see Example 2 page 166 of textbook.

A farmer with 4000 meter of fencing wants to enclose a rectangular plot that is adjacent to a river, as in the graph below. What is the largest area he can fence off this way?