

Chapter 6

Trigonometric functions

6.1 Degrees and radians

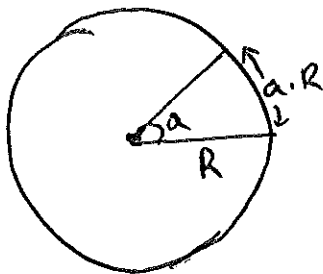
Textbook Section 5.1

There are two major ways of measuring angles in geometry: in *degrees* and in *radians*.

The degree measure was introduced historically in astronomy to measure the displacements of stars, and is based on the fact that there are approximately 360 days in a year (well, there are in fact 365.25 days in a year, but 360 conveniently divides nicely by 2, 3, 4, 6, 10, 12, .., while 365.25 doesn't).

The radian measure is the one more commonly used in mathematics. It is based on the length of portions of a circle:

By definition, the radian measure of an angle is the arc length divided by the radius

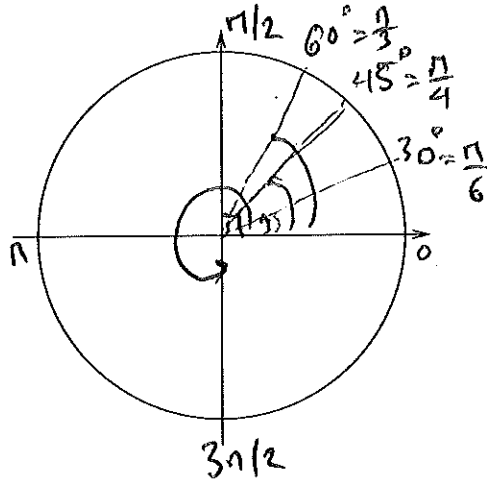


If angle is a
then, the arclength
is $a \cdot R$

→ for full circle:
arclength = $2\pi \cdot R$
with radius = R
→ angle all around is 2π

→ Half-circle:
arclength = $\frac{2\pi R}{2} = \pi \cdot R$
is angle half way around
is π .

Based on this we have the correspondence:



- $90^\circ = \frac{1}{4}$ of circle $= \frac{2\pi}{4} = \frac{\pi}{2}$
- $30^\circ = \frac{1}{12}$ of circle $= \frac{2\pi}{12} = \frac{\pi}{6}$
- $45^\circ = \frac{1}{8}$ of circle $= \frac{2\pi}{8} = \frac{\pi}{4}$
- $60^\circ = \frac{1}{6}$ of circle $= \frac{2\pi}{6} = \frac{\pi}{3}$
- $270^\circ = \frac{3}{4}$ of circle $= 2\pi \cdot \frac{3}{4} = \frac{3\pi}{2}$

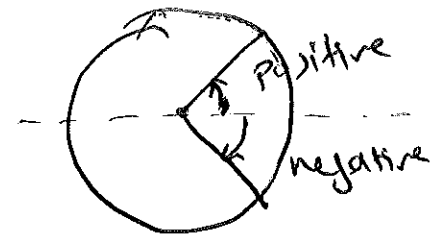
To summarize, to go between radians and degrees and vice-versa,

If x is in degrees & a is in radians:

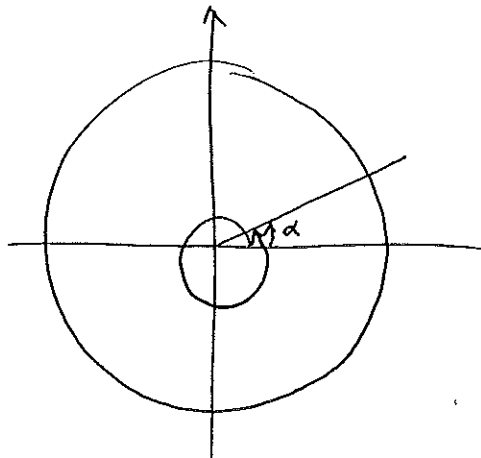
$$a = \frac{2\pi}{360} \cdot x \Rightarrow a = \frac{\pi}{180} x \quad \left\{ \begin{array}{l} x = \frac{360}{2\pi} \cdot a \\ x = \frac{180}{\pi} \cdot a \end{array} \right.$$

By convention, in mathematics we also define a direction to an angle:

- Positive angles go counter clockwise!
- Negative angles go clockwise!



Since the circle wraps around, an angle is always defined up to a value of 2π :



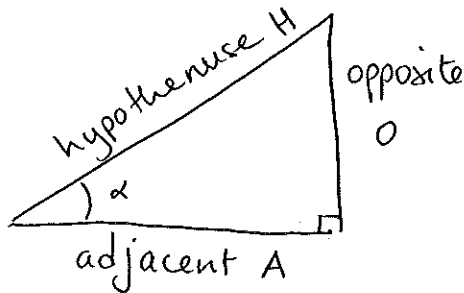
the angle α is the same as the angle $2\pi + \alpha$, which is the same as $4\pi + \alpha$, etc...

→ For any angle α , the actual angle is the same as $\alpha + 2 \cdot \pi \cdot n$ when n is any integer (which could be negative)

6.2 Right-angle triangles and basic trigonometric functions

6.2.1 Sine, cosine and tangent \otimes

Sine, cosine and tangent functions are usually defined through their association with right-angle triangles:



Given an angle α

$$\cos \alpha = \frac{\text{length of adjacent side}}{\text{length of hypotenuse}} = \frac{A}{H}$$

$$\sin \alpha = \frac{\text{length of opposite side}}{\text{length of hypotenuse}} = \frac{O}{H}$$

$$\tan \alpha = \frac{\text{length of opposite side}}{\text{length of adjacent side}} = \frac{O}{A}$$

IMPORTANT CONSEQUENCES: From this diagram, we see that there are two very important formulae relating these three basic trigonometric functions to one another:

• Pythagoras theorem: $A^2 + O^2 = H^2$

But $\cos \alpha = \frac{A}{H} \Rightarrow A = H \cdot \cos \alpha$
 $\sin \alpha = \frac{O}{H} \Rightarrow O = H \cdot \sin \alpha$ } substitute into.

$$(H \cos \alpha)^2 + (H \sin \alpha)^2 = H^2 \Rightarrow H^2 (\cos \alpha)^2 + H^2 (\sin \alpha)^2 = H^2 \quad \otimes$$

$$\Rightarrow H^2 [(\cos \alpha)^2 + (\sin \alpha)^2] = H^2 \Rightarrow \boxed{(\cos \alpha)^2 + (\sin \alpha)^2 = 1}$$

This is true for any angle α !

Example $(\cos(92.5^\circ))^2 + (\sin(92.5^\circ))^2 = 1$

• $\tan \alpha = \frac{O}{A} = \frac{\frac{O}{H}}{\frac{A}{H}} = \frac{\sin \alpha}{\cos \alpha}$ $\tan \alpha = \frac{\sin \alpha}{\cos \alpha}$ \otimes

6.2.2 Co-tangent, secant and cosecant

There are three more important functions to learn, defined as follows:

- $\cotan(\alpha) = \frac{1}{\tan(\alpha)} = \frac{\cos(\alpha)}{\sin(\alpha)}$
- $\sec(\alpha) = \frac{1}{\cos(\alpha)}$
- $\operatorname{cosec}(\alpha) = \frac{1}{\sin(\alpha)}$

These 6 functions altogether form the basic trigonometric functions you will need to know.

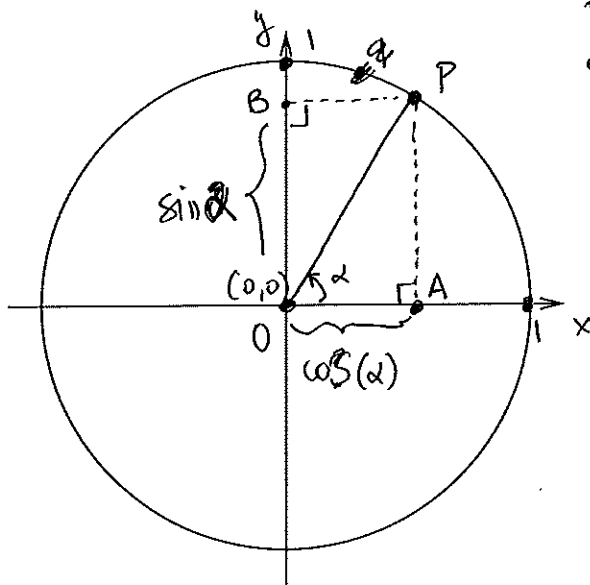
6.3 The unit circle, and the graphs of sine, cosine and tangent

Textbook Section 5.2

6.3.1 Construction of the unit circle

The unit circle is a wonderfully convenient way of *visualizing* the sine and cosine functions.

DEFINITION: The unit circle is a circle of radius 1 centered on $(0,0)$



• On the circle, pick a point P.

• The line between O and P defines an angle α with the horizontal

• Since P is on the circle, so the length OP is 1

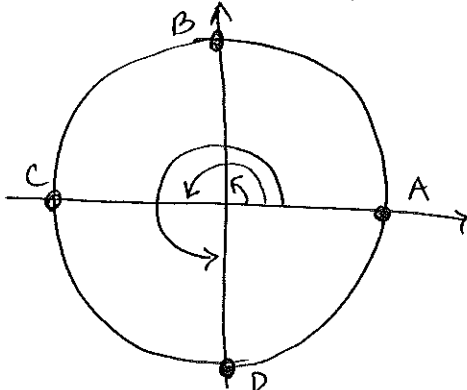
$$\frac{OA}{OP} = \frac{\text{adj}}{\text{hyp}} = \boxed{\cos \alpha = OA}$$

$$\bullet \quad OB = AP$$

$$\frac{AP}{OP} = \frac{\text{opp}}{\text{hyp}} = \boxed{\sin \alpha = AP}$$

\Rightarrow The x-coordinate of P is $\cos \alpha$
The y-coordinate of P is $\sin \alpha$

Based on this, we can already deduce some particular values of the sine, cosine and tangent functions:



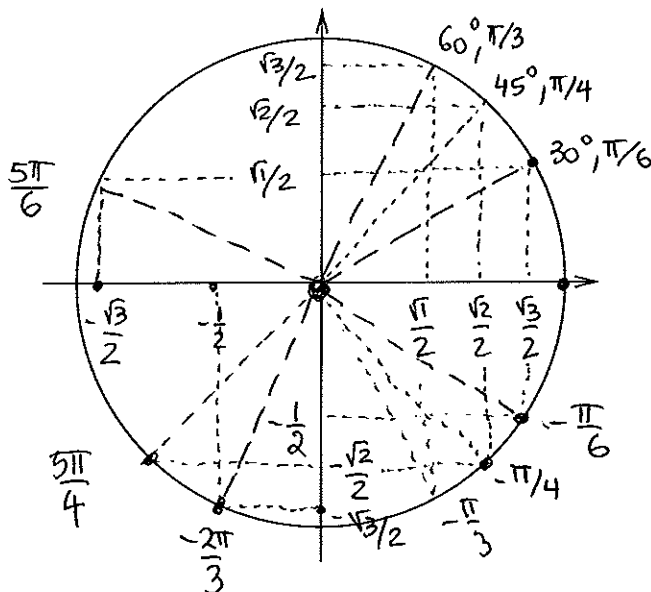
- A: coordinates are (1, 0)
 → A forms angle 0 with horizontal.
 → $\cos 0 = 1$ $\sin 0 = 0$
- B: defines angle $\frac{\pi}{2}$; $\cos \frac{\pi}{2} = 0$; $\sin \frac{\pi}{2} = 1$
- C defines angle π ; $\cos \pi = -1$ $\sin \pi = 0$
- D defines angle $\frac{3\pi}{2}$; $\cos \frac{3\pi}{2} = 0$ $\sin \frac{3\pi}{2} = -1$

6.3.2 Sine and Cosine of important angles

In addition to $\pi/2$, π , $3\pi/2$ and 2π , there are 3 important angles for which you need to know the sine and cosine of:

- 45° ($\pi/4$) \Rightarrow $\sin(\pi/4) = \frac{\sqrt{2}}{2}$ $\cos(\pi/4) = \frac{\sqrt{2}}{2}$
- 60° ($\pi/3$) \Rightarrow $\sin(\pi/3) = \frac{\sqrt{3}}{2}$ $\cos(\pi/3) = \frac{1}{2}$
- 30° ($\pi/6$) \Rightarrow $\sin(\pi/6) = \frac{1}{2}$ $\cos(\pi/6) = \frac{\sqrt{3}}{2}$

Based on the unit circle, we can now find the sine and cosine of many other angles:



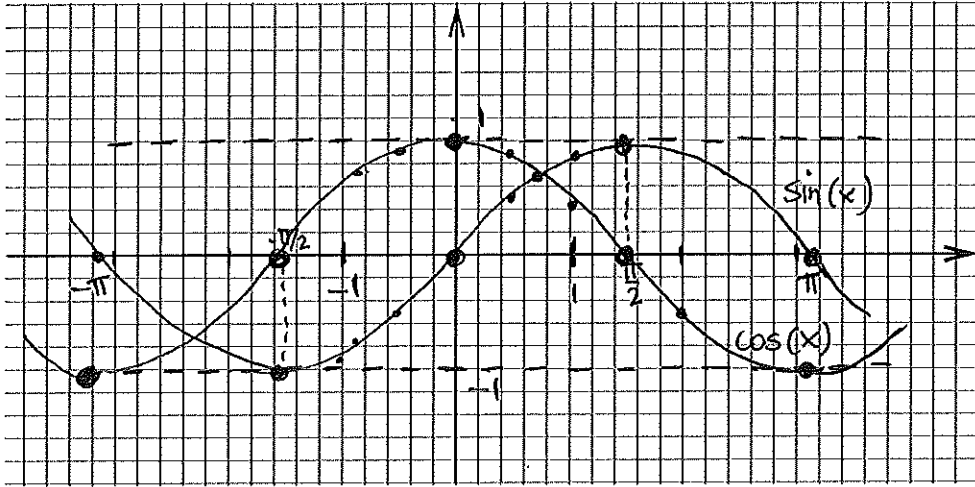
$$\frac{\sqrt{1}}{2} = \frac{1}{2}$$

By symmetry, we can now get any other angle.

- $\sin(-\frac{\pi}{6}) = -\frac{1}{2}$
- $\cos(\frac{5\pi}{6}) = -\frac{\sqrt{3}}{2}$
- $\sin(\frac{5\pi}{4}) = -\frac{\sqrt{2}}{2}$

etc...

Finally, we can use this information to plot the sine and cosine functions:



x	$-\pi$	$-\frac{2\pi}{3}$	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
$\cos x$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	-1
$\sin x$	0	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	0

Note : $\frac{\sqrt{2}}{2} \approx 0.7$ $\frac{\sqrt{3}}{2} \approx 0.85$ $\pi \approx 3$ $\frac{\pi}{2} \approx 1.5$ $\frac{\pi}{4} \approx 0.75 \dots$

6.3.3 What can we deduce from the graphs of $\sin(x)$ and $\cos(x)$? $\frac{\pi}{3} \approx 1$

Based on the graphs of $\sin(x)$ and $\cos(x)$, we see that

- The domain of $\sin x$ and $\cos x$ is \mathbb{R}
- The range of $\sin x$ and $\cos x$ is $[-1, 1]$
- $\cos x$ graph is symmetric about y -axis
 $\rightarrow \cos x$ is even $\rightarrow \cos(-x) = \cos x$
- $\sin x$ graph is point-symmetric about origin
 $\rightarrow \sin x$ is odd $\rightarrow \sin(-x) = -\sin x$

The graphs of $\sin x$ & $\cos x$ are shifted horizontally w.r.t one another

$$\textcircled{\times} \quad \sin(x) = \cos\left(x - \frac{\pi}{2}\right) \quad \cos(x) = \sin\left(x + \frac{\pi}{2}\right)$$