5.5 Applications of exponentials and logarithms

Textbook Section 4.7 and 4.8

5.5.1 Radioactive decay

We saw a few lectures ago that the formula describing the gradual decay of radioactive isotopes that have a life-time T is given by

Using the change-of-base formulas, we can now rewrite this as

In fact, in most textbook you will find that the decay function is given as

where the number r is called the *decay rate*. By comparing the two expressions, note that the half-life T and the decay rate r are related by

EXAMPLE 1: Plutonium-241 has a half-life of 13 years. Express the formula for the amount of Plutonium-241 left after t years both as an exponential in base $1/2$, and an exponential in base e.

Given an initial sample of pure Plutonium-241, what percentage will be left after 100 years?

EXAMPLE 2: Radium-226 is another radioactive element. Its amount in any object decays with time following this function:

$$A(t) = A_0 e^{-0.000427t}$$

where t is expressed in years and A_0 is the original amount. What is the half-life of this element?

5.5.2 Population growth

Let's consider the growth of a rabbit population. Rabbits multiply very fast, and given a suitable ratio of males and females, their population can more-or-less double every month.

1. Suppose that we start we 4 breeding pairs (i.e. 4 males and 4 females). Determine how the number of rabbits N evolves as a function of m, the number of months from now.

2. After how many months does the number of rabbits exceed the human population on Earth (assume it is 6 billion).

3. In practice, what may limit the growth of the rabbit population?

5.5.3 Financial models

Suppose you want to invest \$100,000, and the bank offers you a yearly interest rate of 2%, paid each year on the anniversary of the initial investment. After how many years do you double your initial investment?

Generally speaking, given an interest rate r, how many years does it take to double an investment?

5.6 Solving equations with logarithms and exponentials

Let's now study some additional equations which involve logarithm and exponentials, and yet can be solved analytically.

- Solve the equation $3^{x-1} = 2$

- Find the $x-$ and $y-$intercepts of the function $f(x) = 3^{x+2} - 4$

- Solve the equation $10^{x^2-2x} = 3^x$

- Solve the equation $\log_{10}(4^{x+1}) = \log_2(40)$

- Solve the equation $\ln(x) - \ln(x^2 + 3) = 0$

- Solve the equation $\ln(\ln(x)) = 2$

- Solve the equation $4^x + 2^x - 1 = 0$