

Chapter 1

The notion of functions

Textbook Chapter 1

1.1 The concept of functions

Although the concept of a function was invented a very long time ago, it is very easy today to gain an intuitive notion of what functions are because of their natural role in most computer and/or web-based applications, in engineering, and in economics, etc.

EXAMPLE 1A: ORDERING DIAPERS ON AMAZON

EXAMPLE 1B: BUYING TOMATOES AT SAFEWAY, SELF-CHECKOUT

EXAMPLE 2: BANK ACCOUNTS. CD accounts offer various interest rates depending on the amount you put in: for instance, as of 09/19/15, at Chase, for a 2-year fixed term deposit you get interests of:

- 0.15% per year for accounts under \$10K
- 0.25% per year for accounts between \$10K and under \$100K.

- 0.30% per year for accounts between \$100K and under \$250K.

The banker uses a computer function to tell you what your gain after 2 years will be as a function of your initial investment (cf. <https://www.bankofinternet.com/calculators/apy-interest-calculator>):

EXAMPLE 3: INSTALLING CABLE TV (SEE PROBLEM 22 PAGE 105 OF TEXTBOOK).

1.2 Mathematical definitions

1.2.1 Definition of a function

DEFINITION: A function.

This definition is much more general than what we have seen so far, and covers a much wider class of functions, as for instance in the following examples:

EXAMPLES:

FUNCTIONS DEFINED BY MATHEMATICAL EXPRESSIONS

In most of our work this quarter, however, we will only be considering functions that are defined as *mathematical rules*. These rules take one number, and associate to it another number. The mathematical rule is usually written as:

EXAMPLES:

1.2.2 Functions and their variables

As we saw earlier, mathematical functions are very useful tools to describe how things depend on one another. Usually, when trying to make a mathematical model of a real science or engineering problem, we are trying to understand “How does a quantity y depend on a quantity x ?”. This leads to the following definitions:

DEFINITIONS: Dependent and independent variables

EXAMPLES:

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To evaluate a mathematical function, simply replace the *independent variable* in the mathematical formula by the number considered!

EXAMPLES:

- To find the cost of 10 diaper bags is:
- My gains after a \$200,000 investment are:

- The cost of putting the pole 1 mile away from the box is:

The function can also be applied to expressions instead of numbers. In that case, simply replace the dependent variable by the whole expression

EXAMPLES:

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1.3 Domain of a function

DOMAIN OF DEFINITION OF A FUNCTION: The *Domain of Definition* of a function f consists of all of the values x for which we **are allowed to** or **want to** assign a value $y = f(x)$.

- “**allowed to**” refers to the mathematical rules, i.e. when are you allowed to apply that rule to x

EXAMPLES:

1. $f(x) = \frac{1}{x-1}$

2. $f(x) = \sqrt{x-1}$

3. $f(x) = \frac{\sqrt{2-x}}{x^2-1}$

- “**want to**” refers to the physical problem considered, i.e. what are the values of x that make sense?

Example 1: What is the domain of definitions of the price of diapers?

Example 2: What is the domain of definitions of the cost of installing Cable?

1.5 Functions and their graphs

Graphs are an easy way to *visualize* how things depend on one another, and so they are an ideal *visual* way to represent a function.

DEFINITION: THE GRAPH OF A FUNCTION:

The most basic graphing technique is

- to construct a table of values, with two columns: values of x and corresponding values of $f(x)$
- draw the corresponding points with coordinates $(x, f(x))$.

The more points you have, the more accurate the representation of the function. In most cases, you can join the dots to complete the graph (but be careful of the few counterexamples).

SINGLE VALUE PROPERTY:

- Recall that saying “ f is a function” only makes sense when there is a single value of y corresponding to each value of x . This means that while every function has a graph, not every graph can be the graph of a function!
- THE VERTICAL LINE TEST: A *graph* corresponds to a *function* only if it passes the *Vertical Line Test*: if any vertical line on the graph intersects the line $y = f(x)$ more than once, then f is not Single Valued, therefore f is not a function.

EXAMPLES:

SOME VOCABULARY: A lot more can be learned from a graph. You can immediately see whether

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IMPORTANT NOTE: an empty circle on a graph means that the corresponding point has been “removed” from the graph at this position.

1.6 Techniques in graphing

As we saw, graphs of functions are very useful to get, “at a glance”, some of the basic properties of the functions. However, graphing point by point be a little bit tedious, and there are a lot of faster techniques. They are based on knowing the graphs of “standard” functions, and how mathematical and geometrical manipulations of these graphs relate to one another.

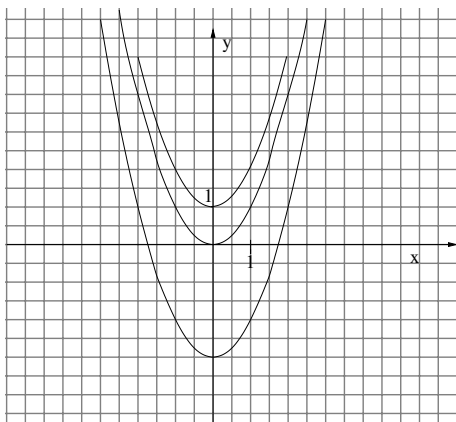
1.6.1 The standard functions

The following standard functions are ones you should know how to graph, accurately, “by heart”, from now on. As the course proceeds, we will continue adding to this list of functions.

1.6.2 Vertical translation

The graph of the function $g(x) = f(x) + a$ can be obtained from that of the function $f(x)$ by translating it vertically by an amount a (downward if $a < 0$ and upward if $a > 0$).

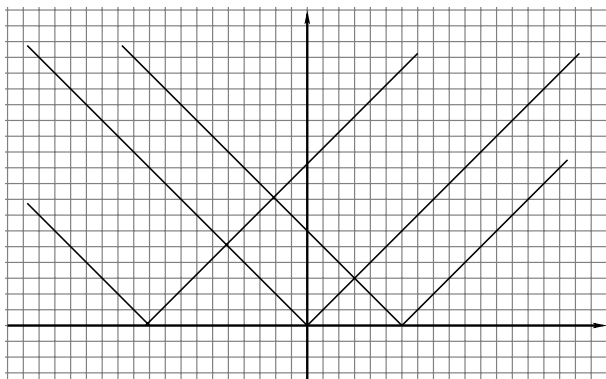
Examples:



1.6.3 Horizontal translation

The graph of the function $g(x) = f(x - a)$ can be obtained from that of the function $f(x)$ by translating it horizontally to the right by an amount a if $a > 0$ and to the left by an amount $|a|$ if $a < 0$.

Examples:



Note:

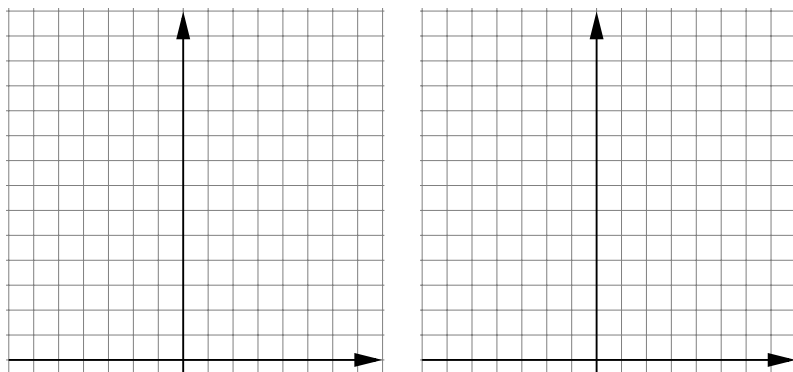
1.6.4 Reflections across the x - and y - axis

The graph of the function $g(x) = -f(x)$ can be obtained from that of the function $f(x)$ by reflection across the x -axis.

The graph of the function $g(x) = f(-x)$ can be obtained from that of the function $f(x)$ by reflection across the y -axis.

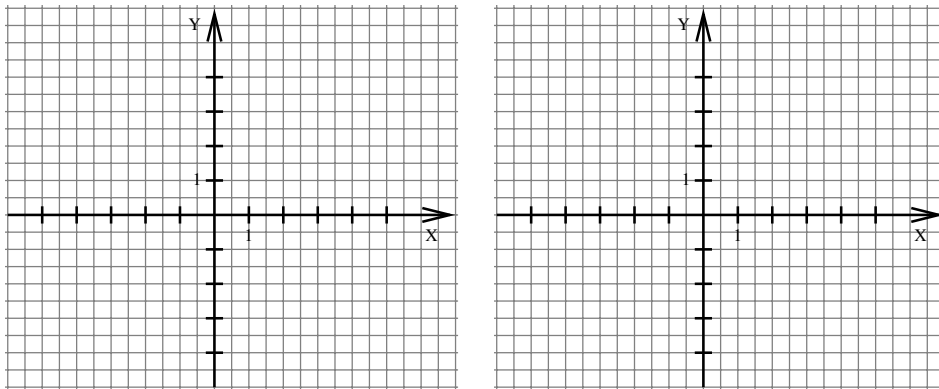
1.6.5 Even and odd functions

Given that the graph of $f(-x)$ can be obtained from that of the function $f(x)$ by reflection across the y -axis (see above), we can deduce an important mathematical property of functions which have graphs that are *symmetric* with respect to the y -axis, as for instance:



DEFINITION:

There is another interesting kind of symmetry, which is a *point symmetry about the origin*. The graphs of these functions look like this:



The functions whose graphs are point symmetric have the property $f(-x) = -f(x)$. Indeed:

DEFINITION:

EXAMPLES: Are these functions odd or even or neither?

- $f(x) = 3x$
- $f(x) = -x + 1$
- $f(x) = x^4$
- $f(x) = -x^5$
- $f(x) = \frac{2}{x}$
- $f(x) = -\frac{4}{x^2}$