5.5   Change of base and applications

Textbook Section 5.4

There are a few manipulations of exponential and logarithms which involve changing bases with the natural base. It is important to master these changes of base formulae.

5.5.1   Changing from base a to the natural exponential

The rule of change of base from a base a exponential to the natural exponential (and vice versa) is:

The reason why this works is simple:

EXAMPLES:

- $2^x =$

- $\left(\frac{1}{4}\right)^x =$

5.5.2   Application of the exponential change of base

Textbook Section 5.7

This change of base is very often used: since the natural base is *the* standard base, most systems which grow or decay exponentially are expressed in base e rather than another base. For example, we saw that for radioactive decay, the amount of radioactive material decays with time as

Another way of writing this exponential decay using a base e exponential is:

In fact, in most textbook you will find that the decay function is given as

where the number r is called the *decay rate*. By comparing the two expressions, note that the half-life τ and the decay rate r are related by

Note that our textbook uses the equivalent formula e^{kt} (where our r is $-k$ in the book).

EXAMPLE 1: Plutonium-241 has a half-life of 13 years. Express the formula for the amount of Plutonium-241 left after t years both as an exponential in base $1/2$, and an exponential in base e.

Given an initial sample of pure Plutonium-241, what percentage will be left after 100 years?

EXAMPLE 2: Radium-226 is another radioactive element. It's amount in any object decays with time following this function:

$$A(t) = A_0 e^{-0.000427t}$$

where t is expressed in years and A_0 is the original amount. What is the half-life of this element?

As another example, we studied the exponential growth of a rabbit population, and assumed that this population doubled every month:

$$R(t) = 2 \times 2^t$$

where t is expressed in (number of) months since the appearance of the first pair of rabbits. After how many months does the number of rabbits exceed the human population on Earth (assume it is 6 billion).

This example also serves to illustrate two more things we need to know how to do:

- Changing from logarithm in base a to natural logarithm
- Solving equations with logarithms and exponentials

5.5.3   Changing from base a to the natural logarithm

The rule of change of base from a base a logarithm to the natural logarithm (and vice versa) is (see result above for example):

The reason why this works is simple too:

EXAMPLES:

- $\log_2(x) =$
- $\log_{\frac{1}{4}} x =$

NOTE: This formula, when applied to a, yields the obvious relationship

If you are not sure of your change-of-base formula, this is a good way of double-checking that the formula you remember is the correct one.

This change of base is particularly useful because most calculators only provide $\ln(x)$ and not $\log_a(x)$. So, whenever you have to calculate $\log_a(x)$, simply evaluate:

EXAMPLE: We already saw some examples earlier, but here are some more...

- Solve the equation $2^x = 6$

- Show that for any a and b, the following is true: $\log_a(b)\log_b(a) = 1$.

5.6   Solving equations with logarithms and exponentials

Let's now study some additional equations which involve logarithm and exponentials, and yet can be solved analytically.

- Solve the equation $3^{x-1} = 2$

- Find the $x-$ and $y-$intercepts of the function $f(x) = 3^{x+2} - 4$

- Solve the equation $10^{x^2 - 2x} = 3^x$

- Solve the equation $\log_{10}(4^{x+1}) = \log_2(40)$

- Solve the equation $\ln(\ln(x)) = 2$

- Solve the equation $4^x + 2^x - 1 = 0$

5.7   Graphing on log-linear and log-log paper

As we saw previously, the graphs of

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Why is this the case? The reason is easy to understand: note that in a log paper the lines are not equally spaced (as in linear paper) but logarithmically spaced. This means that what is plotted is in fact not the number itself, but the logarithm of this number:

As a result,

- for the power law $y = x^a$,

- for an exponential law $y = a^x$,