Chapter 5

Powers, exponentials and logarithms

5.1 Power functions

5.1.1 Definition and examples

Definition:

Examples:

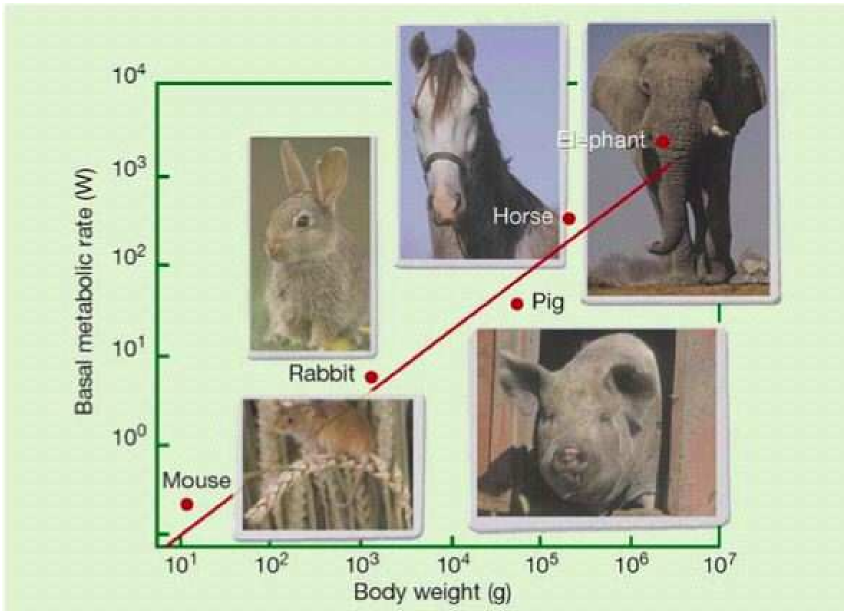
5.1.2 Examples of power functions in Nature

We saw earlier that many geometric problems lead to power-law relationships between variables. However, even in Nature there are many examples of power laws, sometimes with integer exponents, sometimes with non-integer exponents.

- Fundamental forces of Physics follow power-law scalings:
  Example: The force of gravitation

- Allometric laws in Biology: Power functions are frequently found when relating (empirically) two biological variables.

  - Example: Kleiber's Laws:
    Body Metabolism Rate = $3.5(\text{Body Weight})^{3/4}$Watts .



  - The optimal cruising speed for a bird/plane as a function of their body mass:
    Speed = $30\text{Mass}^{1/6}$m/s (mass in kg)

5.1.3 Manipulations of power functions

The following rules apply for manipulating power functions:

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Examples:

5.1.4 The inverse of power functions

RULE:

PROOF 1:

PROOF 2: Remember that to show that two functions (f and g for example) are inverse of one another, you simply evaluate $f \circ g$:

APLICATIONS:

- What is the inverse of $f(x) = x^3$?
- What is the inverse of $f(x) = x^{-1/2}$?
- What is the inverse of $f(x) = x^{-\pi}$?
- Solve for x: $x^{2\pi} - 5x^{\pi} + 6 = 0$

- Find the inverse of $f(x) = \left(\frac{(2x^2)^{3/7}}{x^{1/7}}\right)^{-14}$

5.1.5 Graphs of power functions

Graph on linear paper: The overall shape of the graph of a power function depends on the sign and value of the exponent...

Graph on log-log paper: (see Section)

The graph of a power-law function on log-log paper is always a straight line! We will see in the next section why this is the case. In scientific papers, when researchers want to illustrate that their data supports a power law relationship between two variables, they plot the data on a log-log plot (see the Allometric relationship plot earlier).

5.2 General Exponential functions

Textbook Section 5.1

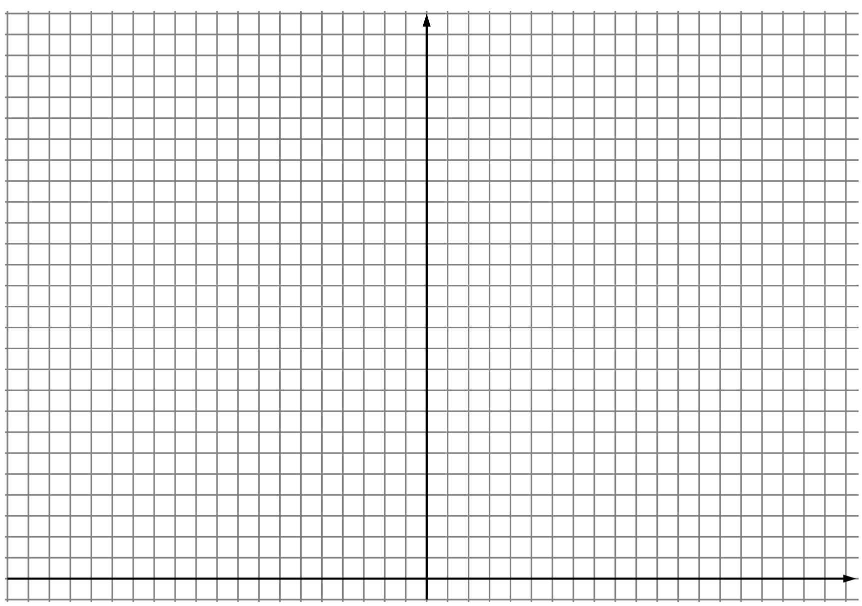
5.2.1 Definition of an exponential functions

DEFINITION:

NOTE: Do not mix up power and exponential functions!

- For power functions:
- For exponential functions:

While we may not be used to thinking of exponents as non-integers, or non-rational numbers, think of the following construction for the function $f(x) = 2^x$:



MANIPULATION OF EXPONENTIAL FUNCTIONS: The rules for manipulating these functions are quite similar to the rules for manipulating powers. Given an exponential function in base a

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Also, given another exponential function in base b

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etc...

EXAMPLES:

- Simplify: $f(x) = \frac{3^{x+2}}{9}$
- Simplify $f(x) = \frac{2^{2x}}{4^x}$
- Simplify $f(x) = 25^x 5^{-x-1}$
- Simplify $f(x) = 2^{2x} 3^x$

5.2.2 Graphs of exponential functions

The graph of an exponential function $f(x) = a^x$ depends on the value of the base a.

Case 1: $a > 1$
Typical example: $f(x) = 2^x$

Case 2: $0 < a < 1$
Typical example: $f(x) = \left(\frac{1}{2}\right)^x$

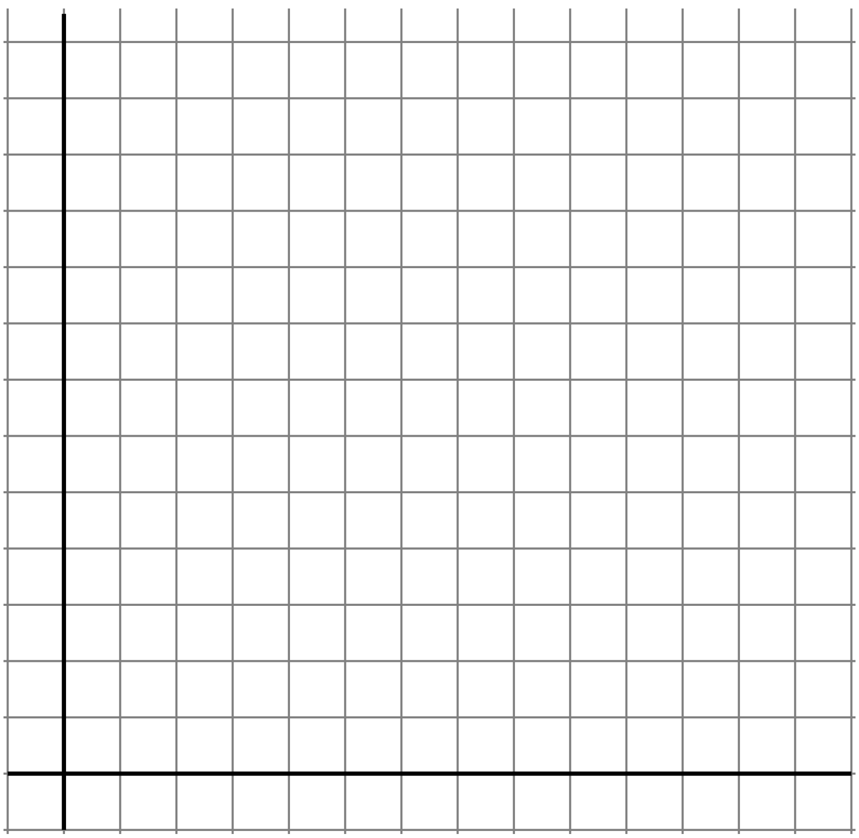
5.2.3 Real-time exponential function

Exponential functions occur in Nature very commonly in systems where something doubles/triples/quadruples... in time at a regular pace (or equivalently gets divided by two/three/four... at a regular pace).

Example of exponential growth:

Exponential growth is also commonly quoted in population dynamics: (see Rabbit example of Week 2).

Example of exponential decay: Everyone in the class takes a coin. We all toss the coin together. At every coin toss, the people who get "tail" stop. People who get "head" continue on. Completing the following table and graph we get:



Exponential decay often occurs in probabilistic systems where there is some chance of "decay" over a certain timescale. The most common example in Nature is radioactive decay. Many atoms are not "stable" atoms: they can, with a certain probability for a given time period, "decay" into another atom (i.e. change into another element). Some elements, however, are more unstable than others, and have a higher probability of decay than others over the same timescale. This propensity to decay is often measured through a "half-life". By definition, the probability of decay is exactly 1/2 after one half-life.

For example:

- The half-life of Carbon 14 (used for radioactive dating) is 5730 years. Carbon 14 decays into Nitrogen 14.
- The half-life of Plutonium 239 (nuclear waste product) is 24,110 years
- The half-life of Iodine 131 (other waste product) is 8 days

The amount of radioactive material left after a certain time is an exponential function of time:

Example 1: Suppose you collect pure Carbon 14 in a sealed jar now.

- What percentage of Carbon 14 is left after 5730 years?
- What percentage of Carbon 14 is left after 100 years?
- What percentage of Carbon 14 is left after a 100,000 years?

Example 2: Suppose you find a sealed jar and identify it contains x percent of Carbon 14 and Nitrogen 14. What do you need to do to find when it was sealed as a function of x?

5.3 General logarithmic functions

Textbook Section 5.3

5.3.1 Definition and graph

Definition:

GRAPH:

Case 1: $a > 1$

Case 2: $0 < a < 1$

DOMAIN OF DEFINITION:

UNIVERSAL PROPERTY OF LOGARITHMS:

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5.3.2 Examples of logarithms in common bases

THE FUNCTION $f(x) = \log_2(x)$ (LOGARITHM IN BASE 2)

THE FUNCTION $f(x) = \log_{10}(x)$ (LOGARITHM IN BASE 10)

5.3.3 The inverse relationships

Since the logarithm in base a is the inverse of the exponential in base a, we have the two fundamental relationships

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These relationships can be used to simplify expressions with exponentials and logs...
EXAMPLES:

- $\log_2(2^x) =$
- $\log_5(5\sqrt{5}) =$

- $\log_{10}(100^x) =$

- $3^{\log_3(2)} =$

- $10^{\log_{100}(2x)} =$

These relationships can also be used to prove important properties of logarithms...

5.3.4 Properties of the logarithms and examples of use

Textbook Section 5.4

The following rules apply for logarithms.

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To show why these formulas are true, we go back to the definition of the logarithm as an inverse, and use the properties of the exponentials: for instance, to show why the first formula is true write:

The other rules are shown in Section.

Examples:

- Combine into one log expression: $\log_2(x^2 - 1) - \log_2(x + 1)$

- Simplify $\log_2(8(x - 2))$:

- Simplify $\log_{10}(100^{x+1}) + \log_{10}\left(\frac{1}{5^x}\right)$

5.4 The natural exponential and the natural logarithm

Textbook Section 5.2

5.4.1 Definition

There is one particular base called the *natural* base for exponentials and logarithm.
DEFINITION:

Remember that e is a real number, with value approximately equal to:

The reason why this peculiar base is important in mathematics will be explored in more detail in Calculus. However, for the moment, just accept the following property of the natural exponential:

Naturally, various function can be constructed from e^x:

Definition:

Properties of the natural logarithm and exponential: since these two functions are inverse of each other...

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5.4.2 The sinh and cosh functions

There are two other important functions based on the natural exponential which are commonly used: they are called hyperbolic sine and hyperbolic cosine functions, and the notations are:

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Note:

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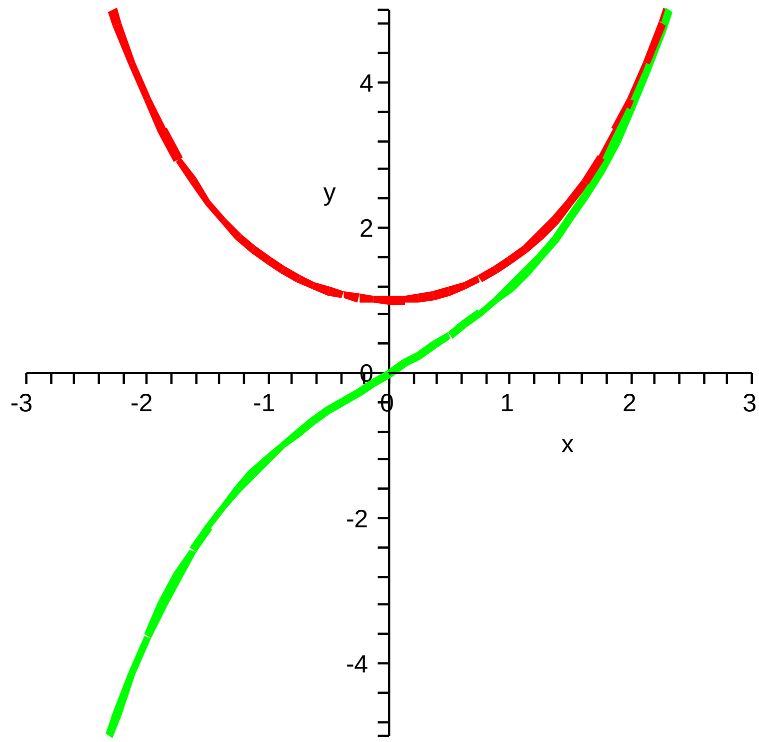
This can easily be seen in their graphs:



To show matematically that cosh is even, note that:

See homework for the case of sinh.

Why are these functions important? These functions will come up a lot in most mathematical models of many systems. However, one rather spectacular property of the cosh function (which can be shown matheamtically) is that its shape is exactly the shape of a chain or rope hanging from two points. That's why it's also sometimes called "The chain function". The "upside-down" version of this function is called a *Catenary*. This shape is used to create arches in building: it can be shown that it is the ideal curve for supporting a lot of weight! This was discovered by Hooke (in 1671) by looking at hanging chains!