Chapter 4

Rational functions and inequalities

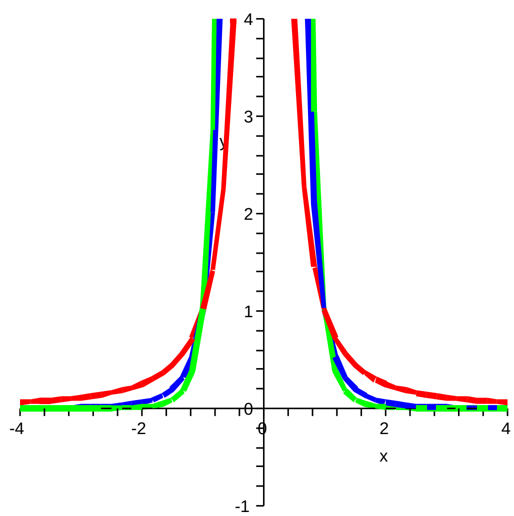
4.1 Rational functions

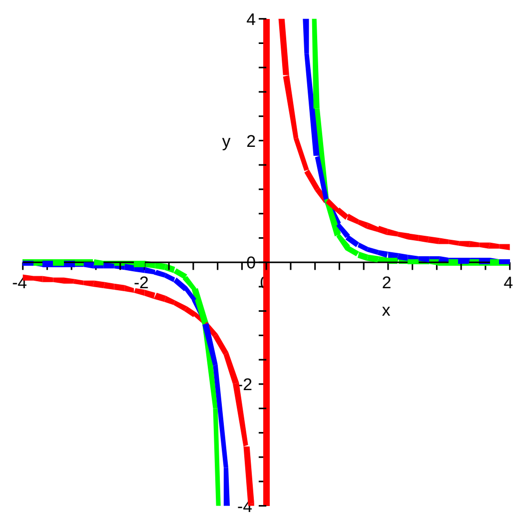
Textbook section 4.7

4.1.1 Basic rational functions and asymptotes

As a first step towards understanding the behavior of rational functions, let's study functions of the kind $f(x) = x^{-n}$, or in other words,

As in the case of power functions, we have two different kinds of behavior depending on whether the power n is even or is odd:





As in the previous case as well, for even powers the function $f(x)$ is even, and for off powers the function $f(x)$ is odd:

NOTE:

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DEFINITION:

DEFINITION:

4.1.2 Other basic rational functions

We saw that the graphs of some functions can be deduced from the graphs of basic functions by various geometrical transformations, such as reflections, vertical and horizontal translations. The same applies here.

EXAMPLE 1: $f(x) = -\frac{1}{x}$

EXAMPLE 2: $f(x) = \frac{1}{x+3}$

Example 3: $f(x) = 3 + \frac{1}{x}$

Example 4: $f(x) = -\frac{1}{(x-2)^2}$

4.1.3 General properties of rational functions

Definitions:

Domain of definition:

Asymptotes vs. excluded points: In most situations, an asymptote occurs when the denominator of the rational function goes to zero. For example, as seen earlier,

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However, it may happen that the asymptote is "canceled out" by a root in the numerator. This situation is easy to determine from the factored form of $f(x)$:

Example 1: $f(x) = \frac{x^2-9}{x+3}$

In that case, the graph of $f(x)$ is the same as the graph of the simplified function *except* that the root/asymptote point has to be removed from the graph since it is not part of the domain of definition. Here,

Example 2: $f(x) = \frac{x-2}{x^2-x-2}$

4.1.4 Studying rational functions using signs tables

Signs tables are extremely useful tools for studying rational functions. They are used in nearly exactly the same way as for polynomial functions:

- Cast the function in a fully factored form, for both the numerator and the denominator. Simplify as needed, but remember the excluded points if there are some.
- Draw the table
- Write **all** the factors vertically on the left, including both the numerator and the denominator.
- Write **all** the points where either the numerator or the denominator goes to 0 on the top, in the correct order. Draw vertical lines below each of them.
- Determine and write the sign of each factor; write zeros where there is a root, and an infinity sign where there is an asymptote.
- Multiply the signs in each interval to determine the sign of the function.

EXAMPLE: $f(x) = \frac{x^2-1}{2x^2+x-2}$

4.1.5 The behavior of rational functions near an asymptote

The behavior of rational functions near an asymptote can be obtained in two completely different ways.

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EXAMPLE: $f(x) = \frac{x^2-1}{2x^2+x-2}$

4.1.6 The behavior of rational functions for large $|x|$

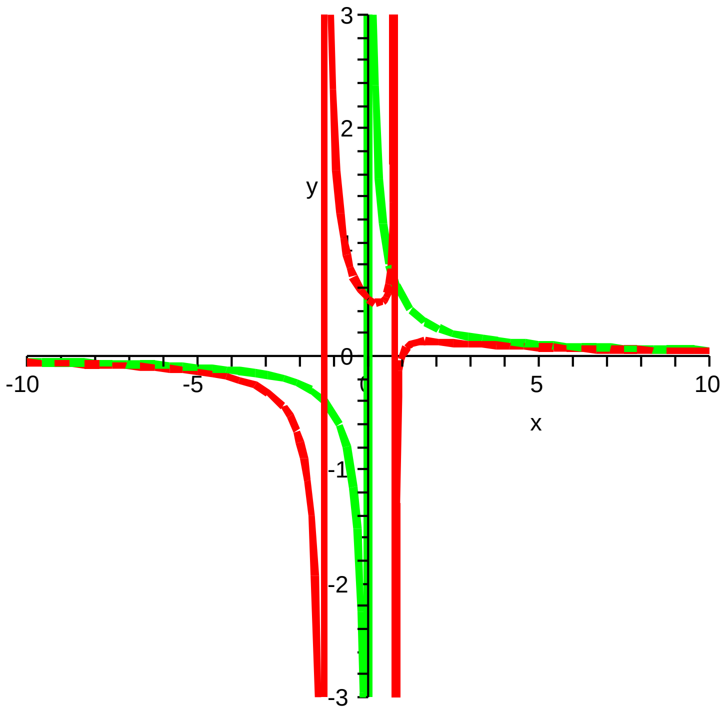
In order to study the behavior of rational functions for large $|x|$ (that is, x going to $+\infty$ or x going to $-\infty$), we use the property learned in the previous chapter about the behavior of *polynomial* functions for large $|x|$:

As a result, for rational functions,

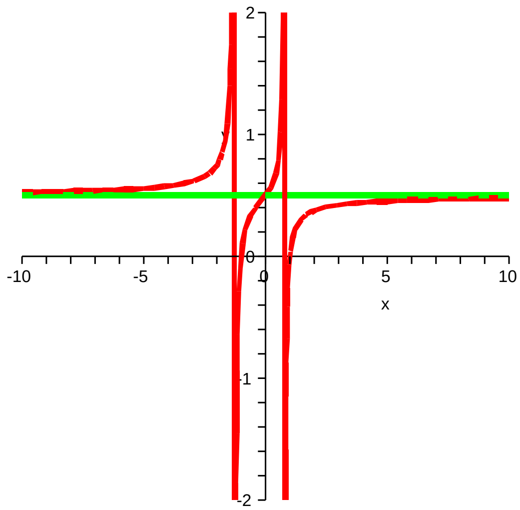
NOTE 1: As in the previous chapter, the behavior for large $|x|$ can easily be used to check the signs table.

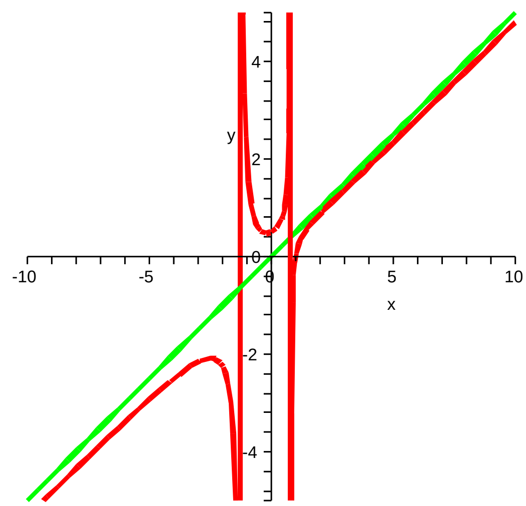
NOTE 2: It is important to realize that not all rational functions go to 0 as x goes to $+\infty$ or x goes to $-\infty$.

EXAMPLE 1: $f(x) = \frac{x-1}{2x^2+x-2}$



EXAMPLE 2: $f(x) = \frac{x^2-1}{2x^2+x-2}$, $g(x) = \frac{x^3-1}{2x^2+x-2}$





Etc, etc...

4.1.7 Example of use of rational functions

We all intuitively know that a square is the shape which minimizes the contour length of a rectangle of a given area.. But why is that?

4.2 Inequalities

Textbook section 2.3

Now that we know a fairly extensive range of basic functions, we have the tools to study inequalities in more detail.

4.2.1 What does "solving" inequalities mean?

Definition:

Examples:

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There are two basic methods for solving inequalities. One relies on knowing the graphs of basic functions and solving equations, the second relies on knowing how to manipulate inequalities to isolate the variable.

4.2.2 "Graphical" method for solving inequalities

A very powerful method for solving inequalities consists in looking at the graphs of the functions $f(x)$ and $g(x)$:

As a result, in order to solve the inequality:

EXAMPLE: Solve the inequality $x^2 + 2x + 1 \geq 1$

EXAMPLE: Solve the inequality $|x + 1| \geq -2x$

Example: Solve the inequality $\frac{3x-2}{x+1} \leq 1$

4.2.3 "Direct" method for solving inequalities

The direct method for solving inequalities relies on isolating the variable x, when possible, on one side of the inequality. Note that this is not always possible, but when it is possible, it can be quite useful. It is also important to know the rules of manipulations of inequalities.

The rules of manipulations of inequalities. The standard rules are the following:

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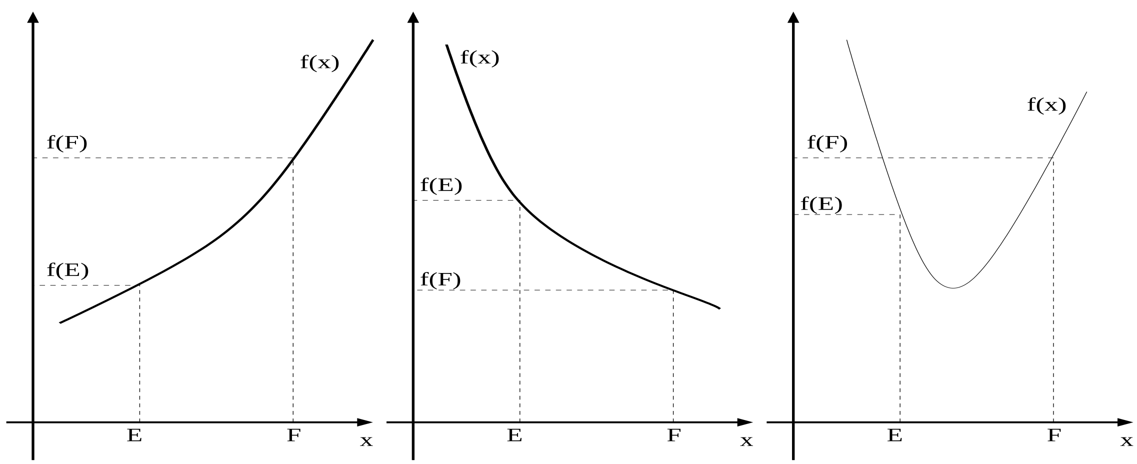
EXAMPLES:

BUT WHERE DO THESE COME FROM, AND HOW TO DEAL WITH MORE COMPLICATED PROBLEMS?

Imagine you have an inequality $E \leq F$ where E and F can be any expressions (numbers, variables, functions, etc..). Then, if you want to transform the inequality with a transformation rule f, note that

- $E \leq F \Rightarrow f(E) \leq f(F)$ if f is an increasing function in $[E, F]$
- $E \leq F \Rightarrow f(E) \geq f(F)$ if f is a decreasing function in $[E, F]$
- We don't know what happens if f has a slope that changes in $[E, F]$.



EXAMPLES:

- You want to add a constant C to both sides of an inequality.

- You want to multiply by a constant C both sides of an inequality.

MORE EXAMPLES: Suppose you have the inequality

$$x \leq y$$

and both x and y are greater than 0. Using the relevant graph, assess whether the following statements are true or false?

- $x^2 \leq y^2$
- $\frac{1}{x} \leq \frac{1}{y}$
- $\sqrt{x} \geq \sqrt{y}$
- $3x + 2 \leq 3y + 2$
- $(x+1)^2 - 5 \leq (y+1)^2 - 5$