Chapter 2

The notion of functions

2.1 The definition of a function

Textbook Section 3.1

2.1.1 The concept of functions

Rough Idea number 1: A function is a rule, which relates things in one group (called a "set" in mathematics) to things in another group. To define a function you need three things:

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Example:

Rough Idea number 2: The function can also be a rule from one set to itself.
Example:

Rough Idea number 3: The rule is a function only if there is a *single* item of the second set associated with each item of the first set.
Example:

Mathematical definition of a function

Examples:

2.1.2 Functions defined by mathematical expressions

In most of our work this quarter, we will be considering functions which are defined as *mathematical rules*. These rules take one number, and associate to it another number.
Examples:

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The mathematical rule is usually written as:

2.1.3 The use of functions

Mathematical functions are very useful tools to describe how things depend on one another. Usually, when trying to make a mathematical model of a real science or engineering problem, we are trying to understand "How does a quantity y depend on a quantity x?". In other words we are trying to find the function f which relates y to x:

Examples:

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Definitions:

To apply the function to a number, simply replace the *independent variable* in the mathematical formula by the number considered!
Examples:

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The function can also be applied to expressions instead of numbers. In that case, simply replace the dependent variable by the whole expression
Examples:

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2.2 Domain of a function

Domain of definition of a function: The *Domain of Definition* of a function f consists of all of the values x for which we **are allowed to** or **want to** assign a value $y = f(x)$.

- **"allowed to"** refers to the mathematical rules, i.e. when are you allowed to apply that rule to x
  Examples:
  1. $f(x) = \frac{1}{x-1}$
  2. $f(x) = \sqrt{x-1}$
- **"want to"** refers to the physical problem considered, i.e. what ar e the values of x which physically make sense?

Example: You have a cylindrical container with total volume $1\mathrm{l} = 1000\mathrm{cm}^3$, and radius 5cm. When filled to a certain height x, the volume contained in the cylinder is $V(x)$. What is the domain of definition of the function $V(x)$?

2.3 Functions and their graphs

Textbook Section 3.2

As we saw in Chapter 1, graphs are an easy way to *visualize* how things depend on one another, and so they are an ideal *visual* way to represent a function.
Definition: the graph of a function:

Single Value Property:

- We saw that saying f is a function only makes sense when there is a single value of y corresponding to each value of x.
- The vertical line test: A *graph* corresponds to a *function* only if is passes the *Vertical Line Test*: if any vertical line on the graph intersects the line $y = f(x)$ more than once, then f is not Single Valued, therefore f is not a function.

A lot more can be learned from a graph. You can immediately see whether

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In this class, we will learn a lot about functions from their graphs, and we will also learn how to draw the graph of standard functions.

2.4 Techniques in graphing

Textbook Section 3.4

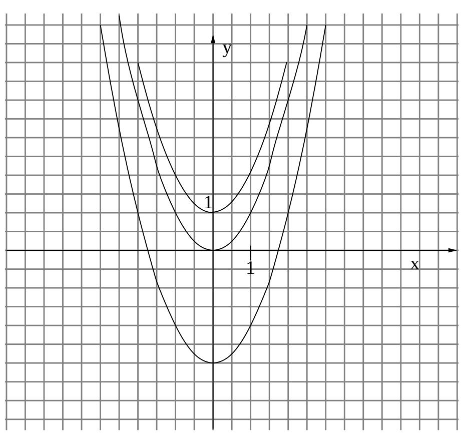
The most basic graphing technique (see Chapter 1) is

- to construct a table of values, with two columns: values of x and corresponding values of $f(x)$
- draw the corresponding points with coordinates $(x, f(x))$.

However, it can be a little bit tedious, and there are a lot of faster techniques. They are based on knowing the graphs of "standard" functions, and how mathematical and geometrical manipulations of these graphs relate to one another.

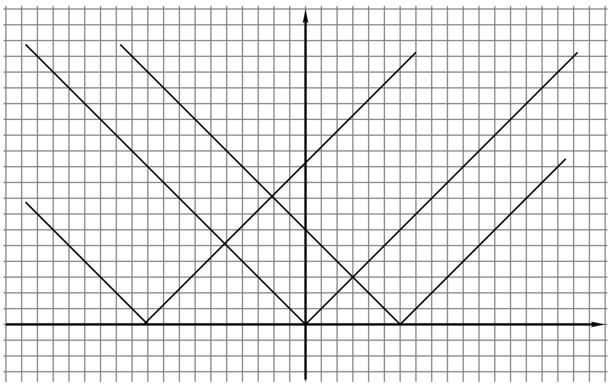
2.4.1 Vertical translation

The graph of the function $g(x) = f(x) + a$ can be obtained from that of the function $f(x)$ by translating it vertically by an amount a (downward if $a < 0$ and upward if $a > 0$).
Examples:



2.4.2 Horizontal translation

The graph of the function $g(x) = f(x - a)$ can be obtained from that of the function $f(x)$ by translating it horizontally to the right by an amount a if $a > 0$ and to the left by an amount $|a|$ if $a < 0$.
Examples:



Note:

2.4.3 Reflections across the x- and y- axis

The graph of the function $g(x) = -f(x)$ can be obtained from that of the function $f(x)$ by reflection across the x-axis.

The graph of the function $g(x) = f(-x)$ can be obtained from that of the function $f(x)$ by reflection across the y-axis.

2.5 Operations with functions

2.5.1 Standard algebraic operations

Functions can be manipulated in the same way as expressions: you can add or substract two functions, multiply or divide them (provided you're not dividing by zero).
Examples:

- Sum of two functions:

- Difference of two functions:

- Product of two functions:

- Ratio of two functions:

However, the following statements are not true in general:

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2.5.2 Composite functions

Physical quantities can depend on each other through a series of causal links: a quantity A depends on a quantity B, and B depends on C, so indirectly, A also depends on C.
Examples:

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To represent this mathematically, we use an *operator* on functions called *composition*.

Examples:

NOTE: Changing the order of the composition yields a different function!

Example: $f(x) = \sqrt{x}$, $g(x) = x^2 + 1$

- $f \circ g$:

- $g \circ f$: