6.5   Trigonometric formulas

There are a few very important formulas in trigonometry, which you will need to know as a preparation for Calculus. These formulas are very useful when trying to simplify complicated expressions which involve trigonometric functions.

6.5.1   The basic formulas

Textbook Section 6.5

We've already seen a few of these formulas: the three basic ones are

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EXAMPLES OF USE: Simplify:

- $1 + \tan^2(x)$

- $1 - \frac{\cot^2(x)}{\csc^2(x)}$

The list of trigonometric identities which can be proved using the basic formulas is endless. See Textbook page 478 for a few of them.

6.5.2   The addition formulas

Textbook Section 8.1 and 8.3

The addition formulas relate the sines and cosines of *sums* of angles to the *products* of sines and cosines of basic angles. The 4 formulas are

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EXAMPLES: These examples show that these formulas indeed work:

- $\cos(\pi/3 - \pi/6)$

- $\cos(\pi/3 + \pi/3)$

- $\sin(\pi/4 + \pi/4)$

Also, they confirm our hunch that $\sin(x)$ and $\cos(x)$ are the "same" function but displaced by $\pi/2$. Indeed,

But where do these formulas come from? Proving the last one will be done in Section, and the others can be deduced from it...

NOTE: There is no equivalent *product* formulas: there is no simple identity for $\cos(ab)$ and $\sin(ab)$. On the other hand, the addition formulas can be used backward to prove the following identities:

Alltogether these identities can be used to prove another nearly-infinite number of identities... for example

$$\frac{\sin(a-b)}{\cos(a)\cos(b)} + \frac{\sin(b-c)}{\cos(b)\cos(c)} + \frac{\sin(c-a)}{\cos(c)\cos(a)} = 0$$

6.5.3   The double-angle formulas

Textbook Section 8.2

As a consequence of the addition formulas, we have 3 more formulas which are called the *double-angle* formulas because they express the sine and cosine of the angle $2a$ in terms of the sine and cosine of the angle a. These formulas are

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To show them, note that

Again, these double-angle formula can be used to simplify trigonometric expressions: for example

$$\frac{\sin(2x)}{\sin(x)} - \frac{\cos(2x)}{\cos(x)} = \sec x$$

Indeed

6.5.4 Summary

Of the various formulas above, only a few have to be known by heart:

- The definition of the various functions:

- The Pytagorean identity:

- The symmetry property:

- The periodicity property:

- The phase-shift property:

- The 2 double-angle formulas:

All of the other formulas can be recovered from combining these ones – so don't clog your memory with formulas... In what follows, we will now use these formulas to solve equations involving trigonometric functions.

6.6   Solving trigonometric equations

Textbook Section 8.4

The formulas studied earlier will now be used to first simplify, and then actually solve trigonometric equations. Let's work by examples.

EXAMPLE 1: Solve the equation $\sin(2x) = \cos(x)$

This example illustrates two steps: using the standard formulas to simplify an equation, and using the unit circle to find how many solutions there are. Another way of doing the second step is to graph the simplified equation:

EXAMPLE 2: Solve the equation $\sin(3x) = \frac{\sqrt{3}}{2}$

This example illustrates that you must be very careful in *not* forgetting the extra solutions coming from the $+2k\pi$ part of the original solution.

EXAMPLE 3: Solve the equation $\cos^2(x) + \cos(x) - 2 = 0$

EXAMPLE 4: Solve the equation $|\sin(x)| = |\cos(x)|$

This equation illustrates the use of the unit circle to solve trigonometric equations and visulalize the solutions...

EXAMPLE 5: However, not all trigonometric equations can easily be solved analytically using these methods. As a last example, let's investigate:

$$3\sin(x) = 1$$

This illustrates the need to introduce inverse trignometric functions

6.7   The inverse trigonometric functions

Textbook Section 8.5

DEFINITIONS: We define the three basic inverse trigonometric functions:

HORIZONTAL LINE TEST? However, note how neither of the three basic trigonometric functions pass the horizontal line test. As a consequence, the domain of definition of the inverse functions is limited to a region where the function does.

For the $\sin(x)$ function and its inverse:

For the $\cos(x)$ function and its inverse:

For the $\tan(x)$ function and its inverse:

As for any function and their inverse, $f(f^{-1}(x)) = f^{-1}(f(x)) = x$ so we have:

What is interesting, however, is to combine the Pythagorean theorem with the inverse equations to get $\sin(\cos^{-1}(x))$ and $\cos(\sin^{-1}(x))$:

These will be used a lot in Calculus!