Handout 1: Algebra Review Worksheet

This worksheet will help you prepare for the Algebra Review. If you make sure you master all of these points, you will do great!

Studying tips: Read each section. If you think you master the material, briefly look over the sample problems to convince yourself you know how to do them (if you're not sure how to do it, then try, and check your answers in the back,otherwise, quickly move on to the next section). If you don't think you master the material, go through the textbook and lecture notes about the material, make sure you understand the examples. Then do as many sample problems as you can, checking your answer in the back. The bottom line is: don't spend too much time on the points you know - make sure you spend most of your time studying for the points you're less familiar with.

Material which is not included in the review test:

- Equations of lines and circles
- Solving equations involving square roots

Finally, if you think there is a typo somewhere, please email me! pgaraud@ams.ucsc.edu

1 Important numbers

See Section 1.1. You need to know the values of a few important numbers:

- $\pi \simeq 3.1$
- $e \simeq 2.7$
- $\sqrt{2} \simeq 1.4$
- $\sqrt{3} \simeq 1.7$

You also need to be able to identify natural numbers, integers, rational numbers and real number.

Sample problems: page 4-5, problems 31-40

2 Intervals

See Section 1.1.

- You need to know the correct notations for intervals, in particular the difference between open and closed intervals. Also be sure to know how to write intervals which extend to either $+\infty$ or $-\infty$.
- You need to be able to write intervals as inequalities, and vice-versa

Example:

- $x > 2 \Leftrightarrow x \in (2, +\infty)$
- $x \leq 2 \Leftrightarrow x \in (-\infty, 2]$

Sample problems: page 4-5, problems 41-52

3 Absolute value

See Section 1.2. The absolute value of an expression E is

- equal to E if E is positive. Examples: $|10| = 10$. $|4-\pi| = 4-\pi$. $|x-1| = x-1$ if $x \geq 1$.
- equal to $-E$ if E is negative (note that if E is negative then $-E$ is positive, as required). Examples: $|-10| = 10 = -(-10)$. $|\pi - 4| = 4-\pi$. $|x-1| = 1-x$ if $x < 1$.

A fundamental property of absolute values is that $|E| = |-E|$ for any expression E: so for example $|10| = |-10|$ and

$$|x-a| = |a-x|$$

for any a, and for example we also have

$$|2x^2 - ax + b| = |-2x^2 + ax - b|$$

Finally, an absolute value can also be seen as a distance: the distance between a point x and a point y is $|x-y|$. So inequalities such as $|x-1| < 2$ means that the distance between x and 1 must be less than 2 (i.e. x is at most 3, and at least -1, or in other words $x \in (-1, 3)$).

Sample problems: page 9-10, problems 15, 23, 29, 33, 35, 41, 47, 59

4 Integer exponents

See Appendix B1. You need to be completely comfortable manipulating expressions with integer powers. This involves *knowing* the following formulae (i.e. knowing where they come from, and how to use them without mistake). For any expression E, and any integer n or m:

$$\begin{aligned}
&E^0 = 1 \\
&E^{n+m} = E^n E^m \\
&E^{-n} = \frac{1}{E^n} \\
&E^{n-m} = \frac{E^n}{E^m} \\
&E^{nm} = (E^n)^m = (E^m)^n
\end{aligned}$$

So for example:

- $100^0 = 1$, $y^0 = 1$, $(2x+1)^0 = 1$, $\left(\frac{1}{x^2-3x-2}\right)^0 = 1$
- $2^2 2^3 = 2^{2+3} = 2^5$, $x^2 x^8 = x^{2+8} = x^{10}$, $(y^2-1)^4 = (y^2-1)(y^2-1)^3$.
- $100^{-2} = \frac{1}{100^2}$, $\frac{1}{x^4} = x^{-4}$, $(4x^2-1)^{-1} = \frac{1}{4x^2-1}$.
- $\frac{2^{15}}{2^2} = 2^{13}$, $\frac{(x^2+1)^3}{(x^2+1)^2} = (x^2+1)^{3-2} = (x^2+1)$, $\frac{(x^2+1)^3}{(x^2+1)^{-2}} = (x^2+1)^{3-(-2)} = (x^2+1)^5$
- $2^6 = (2^3)^2 = 8^2$, $16^2 = (4^2)^2 = 4^4$, $((x-1)^2)^{-2} = (x-1)^{2\times(-2)} = (x-1)^{-4}$

You need to be able to simplify expressions with integer exponents using these formulae. See Appendix B1 for many examples/practise problems.

Note: don't fall into the standard traps:

- $(a^n)^m$ IS NOT a^{n+m}.
- $(a+b)^n$ IS NOT $a^n + b^n$

Sample problems: page A-13 (back of book), problems 19-38

5 Factoring

See Appendix B4 (and 2.1 and 2.2). You need to know how to factor an expression with respect to a particular variable. For this purpose, you should think about this approach to the problem:

- Is the expression one of the standard formulae for factoring?
  1. Is it a difference of squares $a^2 - b^2$? In that case factor as $(a-b)(a+b)$
  2. Is is something like $a^2 + 2ab + b^2$? In that case factor as $(a+b)^2$
  3. Is is something like $a^2 - 2ab + b^2$? In that case factor as $(a-b)^2$
  4. Is it a sum or difference of cubes? In that case use $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$ and $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$
- Is it a quadratic expression of the kind $ax^2 + bx + c$? If that's the case, see the Section on Quadratics.
- If it's not a standard expression, or a quadratic, is there an obvious common factor? If so, begin by factoring it out, and then deal with the next expressions using the same chart.
- If you can't see a common factor, can you group terms in pairs (or sometimes triplets) which can each be factored ? If that's the case, try that, and see if the remaining factors then become a "common factor"

This about covers all the possibilities you will encounter in this class. All it takes now is practise. See Appendix B4 and Sections 2.1 and 2.2 for lots of practise. You should know the standard formulae for factoring *by heart*.

Sample problems: page A-30, problems 1, 11 (use the proper quadratic formula), 19, 21, 39, 55, 59.

6 Fractions

See Appendix B5. You need to be completely comfortable manipulating expressions with fractions. This involves being able to reduce expression to the same denominator, and simplify compound fractions without making any errors.

Reducing to same denominator. Remember that you CANNOT write $\frac{a}{b} + \frac{c}{d} = \frac{a+b}{c+d}$. First you have to find the common denominator for the two fractions. The simplest way is to multiply the first fraction by $\frac{d}{d}$ (which is 1) and the second fraction by $\frac{b}{b}$ (which is also1). Then

$$\frac{a}{b} + \frac{c}{d} = \frac{a}{b}\frac{d}{d} + \frac{c}{d}\frac{b}{b} = \frac{ad}{bd} + \frac{cb}{bd} = \frac{ad+bc}{bd}$$

This is true for any expression a, b, c or d. Examples:

$$\frac{2}{x+3} + \frac{3x}{4-x} = \frac{2}{x+3}\frac{4-x}{4-x} + \frac{3x}{4-x}\frac{x+3}{x+3} = \frac{2(4-x)+3x(x+3)}{(x+3)(4-x)}$$

Sometimes, other common denominators are simpler, as in the example

$$\frac{x}{(x+1)} + \frac{2x}{(x^2-1)}$$

The expression $x^2 - 1$ can be rewritten $(x-1)(x+1)$ and so contains an $x+1$ already. So, to reduce to a common denominator, we write

$$\frac{x}{(x+1)} + \frac{2x}{(x^2-1)} = \frac{x}{(x+1)}\frac{x-1}{x-1} + \frac{2x}{(x^2-1)} = \frac{x(x-1)+2x}{(x-1)(x+1)}$$

Simplifying compound fractions: Compound fractions usually are ratios of fractions or expressions containing fractions. The rule there is

- Step 1: Identify the main fraction "bar". Reduce the expression on top of that to the same denominator, and the expression below that to the same denominator. Example:

$$\frac{\frac{1}{x}+\frac{2}{x+1}}{5-\frac{1}{2x+1}} = \frac{\frac{1}{x}\frac{x+1}{x+1}+\frac{2}{x+1}\frac{x}{x}}{5\frac{2x+1}{2x+1}-\frac{1}{2x+1}} = \frac{\frac{(x+1)+2x}{x(x+1)}}{\frac{5(2x+1)-1}{2x+1}} = \frac{\frac{3x+1}{x(x+1)}}{\frac{10x+4}{2x+1}}$$

- Step 2: use the rule that

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b}\frac{d}{c}$$

(i.e. flip the denominator up and multiply with the numerator). In the example above it becomes

$$\frac{\frac{3x+1}{x(x+1)}}{\frac{10x+4}{2x+1}} = \frac{3x+1}{x(x+1)}\frac{2x+1}{10x+4} = \frac{(3x+1)(2x+1)}{x(x+1)(10x+4)}$$

- Simplify the resulting fraction if possible. In the example above there is no way of simplifying further, but sometimes we may see expression which can be simplified such as

$$\frac{x(x-1)}{2x-2} = \frac{x(x-1)}{2(x-1)} = \frac{x}{2}$$

or

$$\frac{x^2-2}{\sqrt{2}-x} = \frac{(x-\sqrt{2})(x+\sqrt{2})}{\sqrt{2}-x} = -(x+\sqrt{2})$$

since for any expression E, $E/(-E) = -1$, in particular

$$\frac{x-y}{y-x} = -1$$

For practise with these expression and simplification, see Appendix B5.

Finally, remember that under no circumstances are you allowed to simplify fractions which are not factored: for example

$$\frac{x^2+(x+2)}{a+(x+2)} \neq \frac{x^2}{a}$$

So, whenever simplifying rational expressions, always factor the numerator and denominator first! (see factoring section)

Sample problems: page A-35 and A-36: problems 15, 19, 23, 33, 39, 41

7 Quadratic equations

The solutions to the equation $ax^2+bx+c=$ depends on the value of D:

$$D = b^2 - 4ac$$

- if $D < 0$ there are no solutions. The expression ax^2+bx+c cannot be factored.
- if $D = 0$ there is one solution $x_1 = -\frac{b}{2a}$. The expression ax^2+bx+c can be factored as $a(x-x_1)^2$

- if $D > 0$ there are two solutions.

$$x_1 = \frac{-b - \sqrt{D}}{2a} \text{ , } x_2 = \frac{-b + \sqrt{D}}{2a}$$

  The exression $ax^2 + bx + c$ is factored as $a(x - x_1)(x - x_2)$

Examples:

- $x^2 + 6x + 10 = 0$: $D = 6^2 - 4 \times 10 \times 1 = 36 - 40 = -4$. In that case there are no solutions, and the quadratic $x^2 + 6x + 10$ cannot be factored

- $-2x^2 + 12x - 18 = 0$: $D = 12^2 - 4 \times (-18) \times (-2) = 144 - 144 = 0$. In that case there is one solution,

$$x = -\frac{12}{2 \times (-2)} = \frac{12}{4} = 3$$

  and the factored form of the expression is $-2(x - 3)^2$.

- $x^2 - 6x + 4 = 9$: $D = 6^2 - 4 \times 4 \times 1 = 36 - 16 = 20$. In that case there are two solutions,

$$x = \frac{-(-6) + \sqrt{20}}{2 \times (1)} = \frac{6 + 2\sqrt{5}}{2} = 3 + \sqrt{5}, x = \frac{-(-6) - \sqrt{20}}{2 \times (1)} = \frac{6 - 2\sqrt{5}}{2} = 3 - \sqrt{5}$$

  and the factored form of the expression is $\left(x - (3 + \sqrt{5})\right)\left(x - (3 - \sqrt{5})\right) = \left(x - 3 - \sqrt{5}\right)\left(x - 3 + \sqrt{5}\right)$

Sample problems: page 89,problems 43-50 (and factor the quadratics)

8 Higher order equations or other kinds of equations

See Section 2.2. The trick is to reduce them to a "previously solved problem".

8.1 Equations with absolute values

See Section 2.2. For equations with absolute values, note that $|E| = 0$ has two possibilities: $E = 0$ or $-E = 0$. So this reduces to a problem with 2 equations but with no absolute values. You should ALWAYS check if the answer makes sense in the light of the original equations.
Example 1: $|2x + 1| = -x$.

- If $2x + 1 > 0$ then this means $(2x + 1) = -x$ which has the solution $x = -1/3$. When I plug this into the original equation I get $|2 \times (-1/3) + 1| = 1/3$ which works.

- If $2x + 1 < 0$ then this means $-(2x + 1) = -x$ which has the solution $x = -1$. When I plug this into the original equation I get $|2 \times (-1) + 1| = 1$ which works.

Example 2: $|2x - 1| = -x$. This looks very similar BUT:

- If $2x - 1 > 0$ then this means $(2x - 1) = -x$ which has the solution $x = 1/3$. When I plug this into the original equation I get $|2 \times (1/3) + 1| = -1/3$ which doesn't work because an absolute value can't be negative.

- If $2x - 1 < 0$ then this means $-(2x - 1) = -x$ which has the solution $x = 1$. When I plug this into the original equation I get $|2 \times (1) + 1| = -1$ which again doesn't work.

So this time there are no solutions.

Sample problems: page 100-101, problems 1,3,5. See also problems 7 and 9 for a different method.

8.2 Equations which reduce to linear equations

See Section 1.3. Sometimes equations, once manipulated, reduce to a linear equation. Try to see if that's the case for your equation. The trick here is to be absolutely confident of your algebra (i.e. expanding terms out, reducing to the same denominator, etc..). Once you've reduced it to a linear equation then

- The simplest case is when you get $ax = b$ with a non-zero number. Then the solution is $x = b/a$. If $b = 0$, the solution is $x = 0$.
- A tricky case is when you get $ax = b$ with $a = 0$ (i.e. all the x terms dissapear, and you're left with $b = 0$). Then there are two possibilities:
  - if $b = 0$ as well, then the equation is "$0 = 0$" meaning that all possible values of x are a solution. There is "an infinity" of solutions
  - if b is not zero, then the equation $b = 0$ has no solutions, of course (it's an impossibility).

Sample problems: page 17 problems 10, 11, 14, 15, 17, 19, 21.

8.3 Equations which contains higher-order polynomials

For higher-order polynomials always try to factor them first. Then, set each factor to zero to find the solutions. To factor them, use the standard factoring techniques.

Example:

- $x^3 - 2x^2 + 3x - 6 = 0$. Is not a quadratic nor a standard expression. Let's use grouping, then we find $x^2(x - 2) + 3(x - 2) = 0$. We then factor $(x - 2)$ to get $(x - 2)(x^2 + 3) = 0$. The $x^2 + 3$ term can't be factored any further. To find solutions, we then set each factor to 0 in turn: $x - 2 = 0$ implies $x = 2$ is a solution. $x^2 + 3 = 0$ has no solutions (that's why it can't be factored). So there is in total only one solution, $x = 2$.
- $x^4 - 9 = 0$. It's a difference of squares so let's write it as $(x^2 - 3)(x^2 + 3) = 0$. Then note that the first factor is another difference of squares so we factor that again to get $(x - \sqrt{3})(x + \sqrt{3})(x^2 + 3) = 0$. The last factor can't be factored further. To find solutions, we then set each factor to 0 in turn: $x - \sqrt{3} = 0$ implies $x = \sqrt{3}$. $x + \sqrt{3} = 0$ implies $x = -\sqrt{3}$. $x^2 + 3 = 0$ has no solutions (that's why it can't be factored). So there are in total 2 solutions, $x = \sqrt{3}$ and $x = -\sqrt{3}$.

Sample problems: page 101 problem 16, 19, 23, 24.

8.4 Equations which can be reduced to familiar equations by a change of variable

See Section 2.2. For equations which look more complicated, see if a change of variable may help reduce the problem to a "previously solved one" for the new variable. This comes with practise, see textbook and lecture notes for examples.

Sample problems: page 101 problem 27, 43, 45, 47 (only give the exact answer for all these).