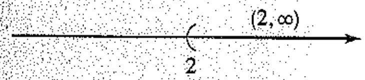
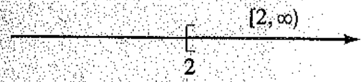
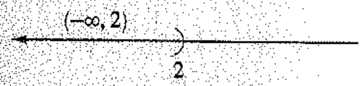
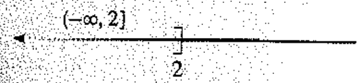
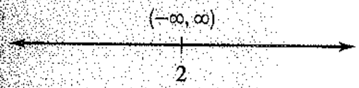
PROPERTY SUMMARY Unbounded Intervals

For a real number a the notations for unbounded intervals are:

Notation	Defining Inequality	Example
(a, ∞)	$x > a$	
$[a, \infty)$	$x \geq a$	
$(-\infty, a)$	$x < a$	
$(-\infty, a]$	$x \leq a$	
$(-\infty, \infty)$		

EXAMPLE 2 Understanding notation for unbounded intervals

Indicate each set of real numbers on a number line:
(a) $(-\infty, 4]$; **(b)** $(-3, \infty)$.

SOLUTION

(a) The interval $(-\infty, 4]$ consists of all real numbers that are less than or equal to 4. See Figure 7.

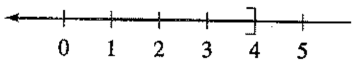


Figure 7

(b) The interval $(-3, \infty)$ consists of all real numbers that are greater than -3. See Figure 8.

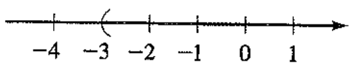


Figure 8

We conclude this section by mentioning that our treatment of the real-number system has been rather informal, and we have not derived any of the rules of arithmetic and algebra using the most basic properties of the real numbers. However, we do list those basic properties and derive some of their consequences in Section A.2 of the Appendix.

EXERCISE SET 1.1

A

In Exercises 1–10, determine whether the number is a natural number, an integer, a rational number, or an irrational number. (Some numbers fit in more than one category.) The following facts will be helpful in some cases: Any number of the form $\sqrt{n}$, where n is a natural number that is not a perfect square, is irrational. Also, the sum, difference, product, and quotient of an irrational number and a nonzero rational are all irrational. (For example, the following four numbers are irrational: $\sqrt{6}$, $\sqrt{10} - 2$, $3\sqrt{15}$, and $-5\sqrt{3}/2$.)

1. **(a)** -203 **(b)** $203/2$

2. **(a)** $27/4$ **(b)** $\sqrt{27/4}$

3. **(a)** 10^6 **(b)** $10^6/10^7$

4. **(a)** 8.7 **(b)** $8.\overline{7}$

5. **(a)** 8.74 **(b)** $8.\overline{74}$

6. **(a)** $\sqrt{99}$ **(b)** $\sqrt{99} + 1$

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are:

(2, ∞)

2

[2, ∞)

2

2

2

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al are all irrational.
are irrational: $\sqrt{6}$,

3. **(a)** 10^6
(b) $10^6/10^7$
6. **(a)** $\sqrt{99}$
(b) $\sqrt{99}+1$

---

7. $3\sqrt{101}+1$ **8.** $(3-\sqrt{2})+(3+\sqrt{2})$
9. $(\sqrt{5}+1)/4$ **10.** $(0.1234)/(0.5677)$

In each of Exercises 11–20, draw a number line similar to the one shown in Figure 1. Then indicate the approximate location of the given number. Where necessary, make use of the approximations $\sqrt{2}\approx 1.4$ *and* $\sqrt{3}\approx 1.7$. *(The symbol* $\approx$ *means* is approximately equal to.*)*

11. $11/4$ **12.** $-7/8$ **13.** $1+\sqrt{2}$
14. $1-\sqrt{2}$ **15.** $\sqrt{2}-1$ **16.** $-\sqrt{2}-1$
17. $\sqrt{2}+\sqrt{3}$ **18.** $\sqrt{2}-\sqrt{3}$ **19.** $(1+\sqrt{2})/2$
20. $(2\sqrt{3}+1)/2$

In Exercises 21–30, draw a number line similar to the one shown in Figure 3(a). Then indicate the approximate location of the given number.

21. $\pi/2$ **22.** $3\pi/2$ **23.** $\pi/6$ **24.** $7\pi/4$
25. -1 **26.** 3 **27.** $\pi/3$ **28.** $3/2$
29. $2\pi+1$ **30.** $2\pi-1$

In Exercises 31–40, say whether the statement is TRUE *or* FALSE. *(In Exercises 37–40, do not use a calculator or table; use instead the approximations* $\sqrt{2}\approx 1.4$ *and* $\pi\approx 3.1$.*)*

31. $-5<-50$ **32.** $0<-1$ **33.** $-2\le -2$
34. $\sqrt{7}-2\ge 0$ **35.** $\frac{13}{14}>\frac{15}{16}$ **36.** $0.\bar{7}>0.7$
37. $2\pi<6$ **38.** $2\le(\pi+1)/2$ **39.** $2\sqrt{2}\ge 2$
40. $\pi^2<12$

In Exercises 41–54, express each interval using inequality notation and show the given interval on a number line.

41. $(2, 5)$ **42.** $(-2, 2)$ **43.** $[1, 4]$
44. $[-\frac{3}{2}, \frac{1}{2}]$ **45.** $[0, 3)$ **46.** $(-4, 0]$
47. $(-3, \infty)$ **48.** $(\sqrt{2}, \infty)$ **49.** $[-1, \infty)$
50. $[0, \infty)$ **51.** $(-\infty, 1)$ **52.** $(-\infty, -2)$
53. $(-\infty, \pi]$ **54.** $(-\infty, \infty)$

B

55. The value of the irrational number π, correct to ten decimal places (without rounding off), is 3.1415926535. By using a calculator, determine to how many decimal places each of the following quantities agrees with π.

(a) $(4/3)^4$: This is the value used for π in the Rhind papyrus, an ancient Babylonian text written about 1650 B.C.

(b) $22/7$: Archimedes (287–212 B.C.) showed that $223/71<\pi<22/7$. The use of the approximation $22/7$ for π was introduced to the Western world through the writings of Boethius (ca. 480–520), a Roman philosopher, mathematician, and statesman. Among all fractions with numerators and denominators less than 100, the fraction $22/7$ is the best approx mation to π.

(c) $355/113$: This approximation of π was obtained in fifth-century China by Zu Chong-Zhi (430–501) and his son. According to David Wells in *The Penguin Dictionary of Curious and Interesting Numbers* (Harmondsworth, Middlesex, England: Viking Penguin, Ltd., 1986), "This is the best approximation of any fraction below 103993/33102."

(d) $\frac{63}{25}\left(\frac{17+15\sqrt{5}}{7+15\sqrt{5}}\right)$: This approximation for π was obtained by the Indian mathematician Scrinivasa Ramanujan (1887–1920).

Remark: A simple approximation that agrees with π through the first 14 decimal places is $\frac{355}{113}\left(1-\frac{0.0003}{3533}\right)$. This approximation was also discovered by Ramanujan. For a fascinating account of the history of π, see the book by Petr Beckmann, *A History of* π, 16th ed. (New York: Barnes & Noble Books, 1989), and for a more modern look at π, see Richard Preston' article, "The Mountains of Pi," in *The New Yorker* (March 2, 1992, pp. 36–67).

C

In Exercises 56–58, give an example of irrational numbers a and b such that the indicated expression is **(a)** *rational and* **(b)** *irrational.*

56. $a+b$ **57.** ab **58.** a/b

59. **(a)** Give an example in which the result of raising a rational number to a rational power is an irrational number.
(b) Give an example in which the result of raising an irrational number to a rational power is a rational number.

60. Can an irrational number raised to an irrational power yield an answer that is rational? This problem shows that the answer is "yes." (However, if you study the following solution very carefully, you'll see that even though we've answered the question in the affirmative, we've not pinpointed the specific case in which an irrational numbe raised to an irrational power is rational.)
(a) Let $A=(\sqrt{2})^{\sqrt{2}}$. Now, either A is rational or A is irr tional. If A is rational, we are done. Why?
(b) If A is irrational, we are done. Why?
Hint: Consider $A^{\sqrt{2}}$.

Remark: For more about this problem and related questions, see the article "Irrational Numbers," by J. P. Jones and S. Toporowski in *American Mathematical Monthly*, vol. 80 (1973), pp. 423–424.

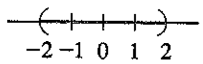

Figure 2
$|x| < 2$

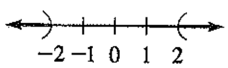

Figure 3
$|x| > 2$

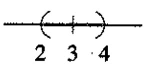

Figure 4
$|x - 3| < 1$

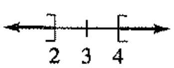

Figure 5
$|x - 3| \ge 1$

SOLUTION

(a) The given inequality tells us that the distance from x to the origin is less than two units. So, as indicated in Figure 2, the number x must lie in the open interval $(-2, 2)$.

(b) The condition $|x| > 2$ means that x is more than two units from the origin. Thus, as indicated in Figure 3, the number x lies either to the right of 2 or to the left of -2.

(c) The given inequality tells us that x must be less than one unit away from 3 on the number line. Looking one unit to either side of 3, then, we see that x must lie between 2 and 4 and x cannot equal 2 or 4. See Figure 4.

(d) The inequality $|x - 3| \ge 1$ says that x is at least one unit away from 3 on the number line. This means that either $x \ge 4$ or $x \le 2$, as shown in Figure 5. [Here's an alternative way of thinking about this: The numbers satisfying the given inequality are precisely those numbers that do *not* satisfy the inequality in part (c). So for part (d), we need to shade that portion of the number line that was not shaded in part (c).]

EXERCISE SET 1.2

A

In Exercises 1–16, evaluate each expression.

1. $|3|$
2. $3 + |-3|$
3. $|-6|$
4. $-6 - |-6|$
5. $|-1 + 3|$
6. $|-6 + 3|$
7. $|-\frac{4}{5}| - \frac{4}{5}$
8. $|\frac{4}{5}| - \frac{4}{5}$
9. $|-6 + 2| - |4|$
10. $|-3 - 4| - |-4|$
11. $||-8| + |-9||$
12. $||-8| - |-9||$
13. $\left|\frac{27 - 5}{5 - 27}\right|$
14. $\frac{|27 - 5|}{|5 - 27|}$
15. $|7(-8)| - |7| \cdot |-8|$
16. $|(-7)^2| + |-7|^2 - (-|-3|)^3$

In Exercises 17–24, evaluate each expression, given that $a = -2$, $b = 3$, and $c = -4$.

17. $|a - b|^2$
18. $a^2 - |bc|$
19. $|c| - |b| - |a|$
20. $|b + c| - |b| - |c|$
21. $|a + b|^2 - |b + c|^2$
22. $\frac{|a| + |b| + |c|}{|a + b + c|}$
23. $\frac{a + b + |a - b|}{2}$
24. $\frac{a + b - |a - b|}{2}$

In Exercises 25–38, rewrite each expression without using absolute value notation.

25. $|\sqrt{2} - 1| - 1$
26. $|1 - \sqrt{2}| + 1$
27. $|x - 3|$ given that $x \ge 3$
28. $|x - 3|$ given that $x < 3$
29. $|t^2 + 1|$
30. $|x^4 + 1|$
31. $|-\sqrt{3} - 4|$
32. $|-\sqrt{3} - \sqrt{5}|$
33. $|x - 3| + |x - 4|$ given that $x < 3$
34. $|x - 3| + |x - 4|$ given that $x > 4$
35. $|x - 3| + |x - 4|$ given that $3 < x < 4$
36. $|x - 3| + |x - 4|$ given that $x = 4$
37. $|x + 1| + 4|x + 3|$ given that $-\frac{5}{2} < x < -\frac{3}{2}$
38. $|x + 1| + 4|x + 3|$ given that $x < -3$

In Exercises 39–48, rewrite each statement using absolute value notation, as in Example 5.

39. The distance between x and 1 is 1/2.
40. The distance between x and 1 is less than 1/2.
41. The distance between x and 1 is at least 1/2.
42. The distance between x and 1 exceeds 1/2.
43. The distance between y and -4 is less than 1.
44. The distance between x^3 and -1 is at most 0.001.
45. The number y is less than three units from the origin.
46. The number y is less than one unit from the number t.
47. The distance between x^2 and a^2 is less than M.
48. The sum of the distances of a and b from the origin is greater than or equal to the distance of $a + b$ from the origin.

In Exercises 49–60, the set of real numbers satisfying the given inequality is one or more intervals on the number line. Show the interval(s) on a number line.

49. $|x| < 4$
50. $|x| < 2$
51. $|x| > 1$
52. $|x| > 0$
53. $|x - 5| < 3$
54. $|x - 4| < 4$
55. $|x - 3| \le 4$
56. $|x - 1| \le \frac{1}{2}$
57. $|x + \frac{1}{3}| < \frac{3}{2}$
58. $|x + \frac{7}{2}| > 1$
59. $|x - 5| \ge 2$
60. $|x + 5| \ge 2$

B

61. In parts (a) and (b), sketch the interval or intervals corresponding to the given inequality:

(a) $|x - 2| < 1$;

(b) $0 < |x - 2| < 1$.

(c) In what way do your answers in (a) and (b) differ? (The distinction is important in the study of *limits* in calculus.)

62. Show that for all real numbers a and b, we have

$$|a| - |b| \leq |a - b|$$

Hint: Beginning with the identity $a = (a - b) + b$, take the absolute value of each side and then use the triangle inequality.

63. Show that

$$|a + b + c| \leq |a| + |b| + |c|$$

for all real numbers a, b, and c. *Hint*: The left-hand side can be written $|a + (b + c)|$. Now use the triangle inequality.

64. Explain why there are no real numbers that satisfy the equation $|x^2 + 4x| = -12$.

C

65. (As background for this exercise, you might want to work Exercise 23.) Prove that

$$\max(a, b) = \frac{a + b + |a - b|}{2}$$

Hint: Consider three separate cases: $a = b$; $a > b$; and $b > a$.

66. (As background for this exercise, you might want to work Exercise 24.) Prove that

$$\min(a, b) = \frac{a + b - |a - b|}{2}$$

67. Complete the following steps to prove the triangle inequality.

(a) Let a and b be real numbers. Which property in the summary box on page 7 tells us that $a \leq |a|$ and $b \leq |b|$?

(b) Add the two inequalities in part (a) to obtain $a + b \leq |a| + |b|$.

(c) In a similar fashion, add the two inequalities $-a \leq |a|$ and $-b \leq |b|$ and deduce that $-(a + b) \leq |a| + |b|$.

(d) Why do the results in parts (b) and (c) imply that $|a + b| \leq |a| + |b|$?

1.3 SOLVING EQUATIONS (REVIEW AND PREVIEW)

I learned algebra fortunately by not learning it at school, and knowing that the whole idea was to find out what x was, and it didn't make any difference how you did it.—Physicist Richard Feynman (1918–1988) in Jagdish Mehra's *The Beat of a Different Drum* (New York: Oxford University Press, 1994)

The title of al-Khwarizmi's second and most important book, Hisab al-jabr w'al muqabala [830]*... has given us the word* algebra. Al-jabr *means transposing a quantity from one side of an equation to the other, while* muqabala *signifies the simplification of the resulting equation.* —Stuart Hollingdale in *Makers of Mathematics* (Harmondsworth, Middlesex, England: Penguin Books, Ltd., 1989)

"Algebra is a merry science," Uncle Jakob would say. "We go hunting for a little animal whose name we don't know, so we call it x. When we bag our game we pounce on it and give it its right name."—Physicist Albert Einstein (1879–1955)

Consider the familiar expression for the area of a circle of radius r, namely, πr^2. Here π is a constant; its value never changes throughout the discussion. On the other hand, r is a variable; we can substitute any positive number for r to obtain the area of a particular circle. More generally, by a **constant** we mean either a particular number (such as π, or -17, or $\sqrt{2}$) or a letter with a value that remains fixed (although perhaps unspecified) throughout a given discussion. In contrast, a **variable** is a letter for which we can substitute any number selected from a given set of numbers. The given set of numbers is called the **domain** of the variable.

Some expressions will make sense only for certain values of the variable. For instance, $1/(x - 3)$ will be undefined when x is 3 (for then the denominator is zero). So in this case we would agree that the domain of the variable x consists of all real numbers except $x = 3$. Similarly, throughout this chapter we adopt the following convention.

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2

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If a_n is not zero, the **degree** of the polynomial equation is the largest exponent of the variable that appears in the equation. For example, the degrees of equations (a), (b), and (c) in the box above are 2, 3, and 4, respectively.

As we've seen in this section, polynomial equations of degree 1 (linear equations) and polynomial equations of degree 2 (quadratic equations) can be solved by using fairly basic algebra. So too can some higher-degree equations. For instance, we can use factoring and the zero-product property to solve equation (b) in the box above. We have

$$x^3 - 2x^2 - 3x = 0$$
$$x(x^2 - 2x - 3) = 0 \qquad \text{factoring out the common factor } x$$
$$x(x - 3)(x + 1) = 0 \qquad \text{factoring the quadratic}$$

Therefore

$$x = 0 \quad \text{or} \quad x - 3 = 0 \quad \text{or} \quad x + 1 = 0 \qquad \text{using the zero-product property}$$

From these last three equations we conclude that the solutions of the third-degree polynomial equation $x^3 - 2x^2 - 3x = 0$ are $x = 0, 3,$ and -1. (You should check for yourself that each of these numbers indeed satisfies the equation.)

Unfortunately, not all polynomial equations are as easy to solve as this last one. Chapter 12 contains a more complete discussion of polynomial equations and an answer to the following question: Is there a general formula, similar to the quadratic formula, for solving *any* polynomial equation?

EXERCISE SET 1.3

A

In Exercises 1–5, determine whether the given value is a solution of the equation.

1. $4x - 5 = -13; x = -2$
2. $\frac{1}{x} = \frac{3}{x} - 1; x = 2$
3. $\frac{2}{y-1} - \frac{3}{y} = \frac{7}{y^2 - y}; y = -3$
4. $(y - 1)(y + 5) = 0; y = 5$
5. $m^2 + m - \frac{5}{16} = 0; m = \frac{1}{4}$
6. Verify that the numbers $1 + \sqrt{5}$ and $1 - \sqrt{5}$ both satisfy the equation $x^2 - 2x - 4 = 0$.

Solve each equation in Exercises 7–19.

7. $2x - 3 = -5$
8. $2m - 1 + 3m + 5 = 6m - 8$
9. $1 - (2m + 5) = -3m$
10. $(x + 2)(x + 1) = x^2 + 11$
11. $t - \{4 - [t - (4 + t)]\} = 6$
12. $\frac{x}{3} + \frac{2x}{5} = \frac{-11}{5}$
13. $1 - \frac{y}{3} = 6$
14. $\frac{x-1}{4} + \frac{2x+3}{-1} = 0$
15. $\frac{1}{x} = \frac{4}{x} - 1$
16. $\frac{1}{y} + 1 = \frac{3}{y} - \frac{1}{2y}$
17. $\frac{1}{x-3} - \frac{2}{x+3} = \frac{1}{x^2 - 9}$
18. $\frac{1}{x-5} + \frac{1}{x+5} = \frac{2x+1}{x^2-25}$
19. $\frac{4}{x+2} + \frac{1}{x-2} = \frac{4}{x^2-4}$
20. $\frac{3}{2x+1} - \frac{4}{x+1} = \frac{2}{2x^2+3x+1}$
21. $\frac{5}{x-4} - \frac{3}{2x^2-5x-12} = \frac{1}{2x+3}$
22. (a) $\frac{2}{3x} = \frac{3}{x}$ (b) $\frac{2}{3x} = \frac{3}{x+1}$ (c) $\frac{2}{3x} = \frac{3}{x} + 1$
23. (a) $\frac{3}{x-2} = \frac{5}{9x}$ (b) $\frac{3}{x-2} = \frac{5}{9x-2}$ (c) $\frac{3}{x-2} = \frac{5}{\frac{5}{3}x - 2}$

In Exercises 24–33, solve each equation by factoring.

24. $x^2 - 5x - 6 = 0$
25. $x^2 - 5x = -6$
26. $10z^2 - 13z - 3 = 0$
27. $3t^2 - t - 4 = 0$
28. $(x + 1)^2 - 4 = 0$
29. $x^2 + 3x - 40 = 0$
30. $x(2x - 13) = -6$
31. $x(3x - 23) = 8$
32. $x(x + 1) = 156$
33. $x^2 + (2\sqrt{5})x + 5 = 0$

In Exercises 34–41, use the quadratic formula to solve each equation. In Exercises 34–39, give two forms for each solution: an expression containing a radical and a calculator approximation rounded off to two decimal places.

34. $2x^2 + 3x - 4 = 0$
35. $4x^2 - 3x - 9 = 0$
36. $x(x + 6) = -2$
37. $x(3x + 8) = -2$

38. $2x^2 - 10 = -\sqrt{2}x$
39. $\sqrt{3}x^2 + \sqrt{3} = 6x$
40. $12x^2 - 25x = -12$
41. $24x^2 + 23x = -5$

In Exercises 42–47, solve the equations using any method you choose.

42. $x^2 = 24$
43. $2y^2 - 50 = 0$
44. $\frac{1}{8} - t^2 = 0$
45. $x^2 - \sqrt{5} = 0$
46. **(a)** $u(u + 18) = -81$
(b) $u(u + 18) = 81$
47. **(a)** $x^2 + 156x + 5963 = 0$
(b) $144y^2 - 54y = 13$
48. Solve each of the following equations for x. *Hint:* As in the text, begin by factoring out a common factor.
(a) $x^3 - 13x^2 + 42x = 0$
(b) $x^3 - 6x^2 + x = 0$

For Exercises 49–58, solve each equation for x in terms of the other letters.

49. $3ax - 2b = b + 3$
50. $ax + b = bx - a$
51. $ax + b = bx + a$
52. $\dfrac{x}{a} + \dfrac{x}{b} = 1$
53. $\dfrac{1}{x} = a + b$
54. $\dfrac{1}{ax} = \dfrac{1}{bx} - \dfrac{1}{c}$
55. $\dfrac{1}{a} - \dfrac{1}{x} = \dfrac{1}{x} - \dfrac{1}{b}$
56. **(a)** $y = mx + b$, where $m \neq 0$
(b) $y - y_1 = m(x - x_1)$, where $m \neq 0$
(c) $\dfrac{x}{a} + \dfrac{y}{b} = 1$
(d) $Ax + By + C = 0$, where $A \neq 0$
57. $(ax + b)^2 - (bx + a)^2 = 0$, where $a \neq \pm b$
58. $(x - p)^2 + (x - q)^2 = p^2 + q^2$

B

In Exercises 59–64, solve each equation for x in terms of the other letters.

59. $a^2(a - x) = b^2(b + x) - 2abx$, where $a \neq b$
60. $\dfrac{b}{ax - 1} - \dfrac{a}{bx - 1} = 0$, where $a \neq b$
61. $\dfrac{a - x}{a - b} - 2 = \dfrac{c - x}{b - c}$
62. $\dfrac{x + 2p}{2q - x} + \dfrac{x - 2p}{2q + x} - \dfrac{4pq}{4q^2 - x^2} = 0$
63. $\dfrac{x - a}{x - b} = \dfrac{b - x}{a - x}$, where $a \neq b$
64. $1 - \dfrac{a}{b}\left(1 - \dfrac{a}{x}\right) - \dfrac{b}{a}\left(1 - \dfrac{b}{x}\right) = 0$

In Exercises 65–68, solve each equation for the indicated variable.

65. $S = 2\pi r^2 + 2\pi rh$; for h
66. $\dfrac{x_1x}{a^2} + \dfrac{y_1y}{b^2} = 1$; for y
67. $d = \dfrac{r}{1 + rt}$; for r
68. $S = \dfrac{rl - a}{r - 1}$; for r

Solve the equations in Exercises 69–74. (In these exercises, you'll need to multiply both sides of the equations by expressions involving the variable. Remember to check your answers in these cases.)

69. $\dfrac{3}{x + 5} + \dfrac{4}{x} = 2$
70. $\dfrac{5}{x + 2} - \dfrac{2x - 1}{5} = 0$
71. $1 - x - \dfrac{2}{6x + 1} = 0$
72. $\dfrac{x^2 - 3x}{x + 1} = \dfrac{4}{x + 1}$
73. $\dfrac{x}{x - 2} + \dfrac{x}{x + 2} = \dfrac{8}{x^2 - 4}$
74. $\dfrac{2x}{x^2 - 1} - \dfrac{1}{x + 3} = 0$
75. Given the equation $\dfrac{1}{x} = \dfrac{1}{a} + \dfrac{1}{b}$:
(a) Solve to show $x = \dfrac{ab}{a + b}$, provided $a + b \neq 0$.
(b) Check the solution.

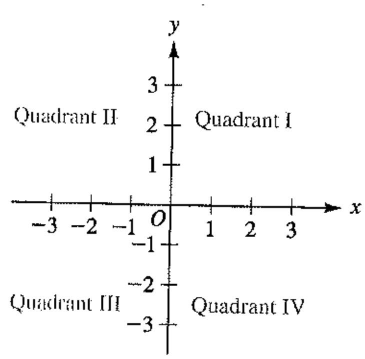

Figure 1

1.4 RECTANGULAR COORDINATES. VISUALIZING DATA

The name coordinate does not appear in the work of Descartes. This term is due to Leibniz and so are abscissa *and* ordinate *(1692).*—David M. Burton in *The History of Mathematics: An Introduction,* 2nd ed. (Dubuque, Iowa: Wm. C. Brown Publishers, 1991)

In previous courses you learned to work with a rectangular coordinate system such as that shown in Figure 1. In this section we review some of the most basic formulas and techniques that are useful here.

The point of intersection of the two perpendicular number lines, or **axes,** is called the **origin** and is denoted by the letter O. The horizontal and vertical axes are often labeled the ***x*-axis** and the ***y*-axis,** respectively, but any other variables will

EXERCISE SET B.4

A

In Exercises 1–66, factor each polynomial or expression. If a polynomial is irreducible, state this. (In Exercises 1–6 the factoring techniques are specified.)

1. (Common factor and difference of squares)
(a) $x^2 - 64$
(b) $7x^4 + 14x^2$
(c) $121z - z^3$
(d) $a^2b^2 - c^2$

2. (Common factor and difference of squares)
(a) $1 - t^4$
(b) $x^6 + x^5 + x^4$
(c) $u^2v^2 - 225$
(d) $81x^4 - x^2$

3. (Trial and error)
(a) $x^2 + 2x - 3$
(b) $x^2 - 2x - 3$
(c) $x^2 - 2x + 3$
(d) $-x^2 + 2x + 3$

4. (Trial and error)
(a) $2x^2 - 7x - 4$
(b) $2x^2 + 7x - 4$
(c) $2x^2 + 7x + 4$
(d) $-2x^2 - 7x + 4$

5. (Sum and difference of cubes)
(a) $x^3 + 1$
(b) $x^3 + 216$
(c) $1000 - 8x^6$
(d) $64a^3x^3 - 125$

6. (Grouping)
(a) $x^4 - 2x^3 + 3x - 6$
(b) $a^2x + bx - a^2z - bz$

7. (a) $144 - x^2$
(b) $144 + x^2$
(c) $144 - (y - 3)^2$

8. (a) $4a^2b^2 + 9c^2$
(b) $4a^2b^2 - 9c^2$
(c) $4a^2b^2 - 9(ab + c)^2$

9. (a) $h^3 - h^5$
(b) $100h^3 - h^5$
(c) $100(h + 1)^3 - (h + 1)^5$

10. (a) $x^4 - x^2$
(b) $3x^4 - 48x^2$
(c) $3(x + h)^4 - 48(x + h)^2$

11. (a) $x^2 - 13x + 40$
(b) $x^2 - 13x - 40$

12. (a) $x^2 + 10x + 16$
(b) $x^2 - 10x + 16$

13. (a) $x^2 + 5x - 36$
(b) $x^2 - 13x + 36$

14. (a) $x^2 - x + 6$
(b) $x^2 + x - 6$

15. (a) $3x^2 - 22x - 16$
(b) $3x^2 - x - 16$

16. (a) $x^2 - 4x + 1$
(b) $x^2 - 4x - 3$

17. (a) $6x^2 + 13x - 5$
(b) $6x^2 - x - 5$

18. (a) $16x^2 + 18x - 9$
(b) $16x^2 - 143x - 9$

19. (a) $t^4 + 2t^2 + 1$
(b) $t^4 - 2t^2 + 1$
(c) $t^4 - 2t^2 - 1$

20. (a) $t^4 - 9t^2 + 20$
(b) $t^4 - 19t^2 + 20$
(c) $t^4 - 19t^2 - 20$

21. (a) $4x^3 - 20x^2 - 25x$
(b) $4x^3 - 20x^2 + 25x$

22. (a) $x^3 + x^2 + x$
(b) $x^3 + 2x^2 + x$

23. (a) $ab - bc + a^2 - ac$
(b) $(u + v)x - xy + (u + v)^2 - (u + v)y$

24. (a) $3(x + 5)^3 + 2(x + 5)^2$
(b) $a(x + 5)^3 + b(x + 5)^2$

25. $x^2z^2 + xzt + xyz + yt$

26. $a^2t^2 + b^2t^2 - cb^2 - ca^2$

27. $a^4 - 4a^2b^2c^2 + 4b^4c^4$

28. $A^2 - B^2 + 16A + 64$

29. $A^2 + B^2$

30. $x^2 + 64$

31. $x^3 + 64$

32. $27 - (a - b)^3$

33. $(x + y)^3 - y^3$

34. $(a + b)^3 - 8c^3$

35. $x^3 - y^3 + x - y$

36. $8a^3 + 27b^3 + 2a + 3b$

37. (a) $p^4 - 1$
(b) $p^8 - 1$

38. (a) $p^4 + 4$
(b) $p^4 - 4$

39. $x^3 + 3x^2 + 3x + 1$

40. $-1 + 6x - 12x^2 + 8x^3$

41. $x^2 + 16y^2$

42. $4u^2 + 25v^2$

43. $\frac{25}{16} - c^2$

44. $\frac{81}{4} - y^2$

45. $z^4 - \frac{81}{16}$

46. $\frac{(a + b)^2}{4} - \frac{a^2b^2}{9}$

47. $\frac{125}{m^3n^3} - 1$

48. $\frac{x^3}{8} - \frac{512}{x^3}$

49. $\frac{1}{4}x^2 + xy + y^2$

50. $x^2 + x + 1$

51. $64(x - a)^3 - x + a$

52. $64(x - a)^4 - x + a$

53. $x^2 - a^2 + y^2 - 2xy$

54. $a^4 - (b + c)^4$

55. $21x^3 + 82x^2 - 39x$

56. $x^3a^2 - 8y^3a^2 - 4x^3b^2 + 32y^3b^2$

57. $12xy + 25 - 4x^2 - 9y^2$

58. $ax^2 + (1 + ab)xy + by^2$

59. $ax^2 + (a + b)x + b$

60. $(5a^2 - 11a + 10)^2 - (4a^2 - 15a + 6)^2$

61. $(x + 1)^{1/2} - (x + 1)^{3/2}$

62. $(x^2 + 1)^{3/2} + (x^2 + 1)^{7/2}$

63. $(x + 1)^{-1/2} - (x + 1)^{-3/2}$

64. $(x^2 + 1)^{-2/3} + (x^2 + 1)^{-5/3}$

65. $(2x + 3)^{1/2} - \frac{1}{3}(2x + 3)^{3/2}$

66. $(ax + b)^{-1/2} - \sqrt{ax + b}/b$

In Exercises 67 and 68, evaluate the given expressions using factoring techniques. (The point here is to do as little actual arithmetic as possible.)

67. (a) $100^2 - 99^2$
(b) $8^3 - 6^3$
(c) $1000^2 - 999^2$

68. (a) $10^3 - 9^3$
(b) $50^2 - 49^2$
(c) $\frac{15^3 - 10^3}{15^2 - 10^2}$

B

In Exercises 69–73, factor each expression.

69. $A^3 + B^3 + 3AB(A + B)$

70. $p^3 - q^3 - p(p^2 - q^2) + q(p - q)^2$

71. $2x(a^2 + x^2)^{-1/2} - x^3(a^2 + x^2)^{-3/2}$

72. $\frac{1}{2}(x - a)^{-1/2}(x + a)^{-1/2} - \frac{1}{2}(x + a)^{1/2}(x - a)^{-3/2}$

73. $y^4 - (p + q)y^3 + (p^2q + pq^2)y - p^2q^2$

74. (a) Factor $x^4 + 2x^2y^2 + y^4$.
(b) Factor $x^4 + x^2y^2 + y^4$. *Hint:* Add and subtract a term. [Keep part (a) in mind.]
(c) Factor $x^6 - y^6$ as a difference of squares.
(d) Factor $x^6 - y^6$ as a difference of cubes. [Use the result in part (b) to obtain the same answer as in part (c).]

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us.)

EXAMPLE 8 Simplifying a compound fraction

Simplify: $\dfrac{x - \dfrac{1}{x^2}}{\dfrac{1}{x^2} - 1}$.

SOLUTION

$$\frac{x - \dfrac{1}{x^2}}{\dfrac{1}{x^2} - 1} = \frac{x^2}{x^2} \cdot \frac{x - \dfrac{1}{x^2}}{\dfrac{1}{x^2} - 1} \qquad \text{multiplying by } 1 = \frac{x^2}{x^2}$$

$$= \frac{x^3 - 1}{1 - x^2} = -\frac{x^3 - 1}{x^2 - 1}$$

$$= -\frac{(x - 1)(x^2 + x + 1)}{(x - 1)(x + 1)} = -\frac{x^2 + x + 1}{x + 1}$$

EXERCISE SET B.5

A

In Exercises 1–12, reduce the fractions to lowest terms.

1. $\dfrac{x^2 - 9}{x + 3}$

2. $\dfrac{25 - x^2}{x - 5}$

3. $\dfrac{x + 2}{x^4 - 16}$

4. $\dfrac{x^2 - x - 20}{2x^2 + 7x - 4}$

5. $\dfrac{x^2 + 2x + 4}{x^3 - 8}$

6. $\dfrac{a + b}{ax^2 + bx^2}$

7. $\dfrac{9ab - 12b^2}{6a^2 - 8ab}$

8. $\dfrac{a^3b^2 - 27b^5}{(ab - 3b^2)^2}$

9. $\dfrac{a^3 + a^2 + a + 1}{a^2 - 1}$

10. $\dfrac{(x - y)^2(a + b)}{(x^2 - y^2)(a^2 + 2ab + b^2)}$

11. $\dfrac{x^3 - y^3}{(x - y)^3}$

12. $\dfrac{x^4 - y^4}{(x^4y + x^2y^3 + x^3y^2 + xy^4)(x - y)^2}$

In Exercises 13–55, carry out the indicated operations and simplify where possible.

13. $\dfrac{2}{x - 2} \cdot \dfrac{x^2 - 4}{x + 2}$

14. $\dfrac{ax + 3}{2a + 1} \div \dfrac{a^2x^2 + 3ax}{4a^2 - 1}$

15. $\dfrac{x^2 - x - 2}{x^2 + x - 12} \cdot \dfrac{x^2 - 3x}{x^2 - 4x + 4}$

16. $(3t^2 + 4tx + x^2) \div \dfrac{3t^2 - 2tx - x^2}{t^2 - x^2}$

17. $\dfrac{x^3 + y^3}{x^2 - 4xy + 3y^2} \div \dfrac{(x + y)^3}{x^2 - 2xy - 3y^2}$

18. $\dfrac{a^2 - a - 42}{a^4 + 216a} \div \dfrac{a^2 - 49}{a^3 - 6a^2 + 36a}$

19. $\dfrac{4}{x} - \dfrac{2}{x^2}$

20. $\dfrac{1}{3x} + \dfrac{1}{5x^2} - \dfrac{1}{30x^3}$

21. $\dfrac{6}{a} - \dfrac{a}{6}$

22. $\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c}$

23. $\dfrac{1}{x + 3} + \dfrac{3}{x + 2}$

24. $\dfrac{4}{x - 4} - \dfrac{4}{x + 1}$

25. $\dfrac{3x}{x - 2} - \dfrac{6}{x^2 - 4}$

26. $1 + \dfrac{1}{x} - \dfrac{1}{x^2}$

27. $\dfrac{a}{x - 1} + \dfrac{2ax}{(x - 1)^2} + \dfrac{3ax^2}{(x - 1)^3}$

28. $\dfrac{a^2 + 5a - 4}{a^2 - 16} - \dfrac{2a}{2a^2 + 8a}$

29. $\dfrac{4}{x - 5} - \dfrac{4}{5 - x}$

30. $\dfrac{x}{x + a} + \dfrac{a}{a - x}$

31. $\dfrac{a^2 + b^2}{a^2 - b^2} + \dfrac{a}{a + b} + \dfrac{b}{b - a}$

32. $\dfrac{3}{2x + 2} - \dfrac{5}{x^2 - 1} + \dfrac{1}{x + 1}$

33. $\dfrac{1}{x^2 + x - 20} - \dfrac{1}{x^2 - 8x + 16}$

34. $\dfrac{4}{6x^2 + 5x - 4} + \dfrac{1}{3x^2 + 4x} - \dfrac{1}{2x - 1}$

35. $\dfrac{2q + p}{2p^2 - 9pq - 5q^2} - \dfrac{p + q}{p^2 - 5pq}$

36. $\dfrac{1}{x - 1} + \dfrac{1}{x^2 - 1} + \dfrac{1}{x^3 - 1}$

37. $\dfrac{\frac{1}{x}+1}{\frac{1}{x}-1}$

38. $\dfrac{\frac{4}{a}-a}{\frac{2}{a}+1}$

39. $\dfrac{\frac{1}{x}-\frac{1}{a}}{x-a}$

40. $\dfrac{\frac{1}{a}+\frac{1}{b}}{\frac{1}{a}-\frac{1}{b}}$

41. $\dfrac{a-\frac{1}{a}}{1+\frac{1}{a}}$

42. $\dfrac{\frac{1}{x^2}-\frac{1}{y^2}}{\frac{1}{x}+\frac{1}{y}}$

43. $\dfrac{\frac{1}{2+h}-\frac{1}{2}}{h}$

44. $\dfrac{\frac{3}{x^2+h}-\frac{3}{x^2}}{h}$

45. $\dfrac{\frac{a}{x^2}+\frac{x}{a^2}}{a^2-ax+x^2}$

46. $\dfrac{x+\frac{xy}{y-x}}{\frac{y^2}{x^2-y^2}+1}$

47. $(x^{-1}+2)^{-1}$

48. $\dfrac{(x^{-2}+2x)^{-1}}{x^2}$

49. $\left(\dfrac{1}{a^{-1}}+\dfrac{1}{a^{-2}}\right)^{-1}$

50. $(a^{-1}+a^{-2})^{-1}$

51. $x(x+y)^{-1}+y(x-y)^{-1}$

52. $x(2x-2y)^{-1}+y(2y-2x)^{-1}$

B

53. $\dfrac{\frac{a+b}{a-b}+\frac{a-b}{a+b}}{\frac{a-b}{a+b}-\frac{a+b}{a-b}}\cdot\dfrac{ab^3-a^3b}{a^2+b^2}$

54. $\dfrac{\sqrt{x+a}}{\sqrt{x-a}}-\dfrac{\sqrt{x-a}}{\sqrt{x+a}}$

C

55. $\dfrac{\left(a+\frac{1}{b}\right)^a\left(a-\frac{1}{b}\right)^b}{\left(b+\frac{1}{a}\right)^a\left(b-\frac{1}{a}\right)^b}$

56. Consider the three fractions

$$\frac{b-c}{1+bc},\qquad \frac{c-a}{1+ca},\qquad \text{and}\qquad \frac{a-b}{1+ab}$$

(a) If $a=1$, $b=2$, and $c=3$, find the sum of the three fractions. Also compute their product. What do you observe?

(b) Show that the sum and the product of the three given fractions are, in fact, always equal.