9.3 A Fourier view of homogeneous isotropic turbulence

With the advent of supercomputing, Direct Numerical Simulations in triply-periodic domains (which are inherently homogeneous) rapidly became one of the preferred approaches to study turbulence. As we shall demonstrate, the two views of the problem give equivalent results, as expected. But the Fourier representation is arguably more directly accessible, the main advantage being that the entire flow can be computed and therefore known at any point in time and space (given enough computational power). In addition, many of the quantities introduced in the previous section can be written in a very simple form in terms of the Fourier expansion of the velocity field.

9.3.1 Definitions

The triply-periodic nature of the system naturally lends itself to the use of a Fourier decomposition. Assuming that the simulation is performed in a cubic box of size L, the periodicity of the flow in L in each direction suggests the decomposition

$$q(\boldsymbol{x},t) = \sum_{\boldsymbol{k}} \hat{q}(\boldsymbol{k},t) \exp(i\boldsymbol{k}\cdot\boldsymbol{x}) \tag{9.65}$$

where $\boldsymbol{k}$ is the 3D wavevector $\boldsymbol{k} = (k_x, k_y, k_z) = \frac{2\pi}{L}(n_x, n_y, n_z)$ and n_x, n_y, n_z are integers, and where

$$\hat{q}(\boldsymbol{k},t) = \frac{1}{L^3}\int_0^L \int_0^L \int_0^L q(\boldsymbol{x},t)\exp(-i\boldsymbol{k}\cdot\boldsymbol{x})dxdydz \equiv \langle q(\boldsymbol{x},t)\exp(-i\boldsymbol{k}\cdot\boldsymbol{x})\rangle_L \tag{9.66}$$

where $\langle\cdot\rangle_L$ denotes a volume average over the cube. To show this, one must make use of the orthogonality property that

$$\langle \exp(i\boldsymbol{k}\cdot\boldsymbol{x})\exp(-i\boldsymbol{k}'\cdot\boldsymbol{x})\rangle_L = \delta_{\boldsymbol{k},\boldsymbol{k}'} \tag{9.67}$$

that is, 0 if $\boldsymbol{k} \neq \boldsymbol{k}'$ and 1 otherwise. It is easy to show that this Fourier representation has the property

$$q(\boldsymbol{k},t) = q^*(-\boldsymbol{k},t) \tag{9.68}$$

when $q(\boldsymbol{x},t)$ is real.

9.3.2 What happened to isotropy?

At this point, one may question whether a triply periodic box is appropriate to describe an isotropic system – and indeed, it is not, since by construction the box knows about preferred directions (i.e. $\boldsymbol{e}_x$, $\boldsymbol{e}_y$ and $\boldsymbol{e}_z$). Furthermore, the wavevectors $\boldsymbol{k}$ being discretely distributed, they cannot represent all possible directions of the flow, since they exist on a grid (see Figure 9.4). Note, however,

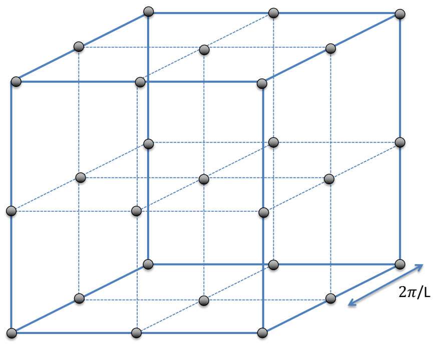

Figure 9.4: Discretization of wavenumber space.

that the grid has a spacing of $2\pi/L$ in each direction, so by taking the limit $L \to \infty$, we ultimately create a denser and denser grid, for which isotropy gradually becomes a meaningful concept. In all that follows, we will therefore always think of taking the limit $L \to \infty$ when thinking of isotropy.

9.3.3 The incompressibility condition

The Fourier expansion can be applied to each component of the velocity field individually, which then defines

$$\boldsymbol{u}(\boldsymbol{x}, t) = \sum_{\boldsymbol{k}} \hat{\boldsymbol{u}}(\boldsymbol{k}, t) \exp(i\boldsymbol{k} \cdot \boldsymbol{x}) \tag{9.69}$$

The incompressibility condition requires

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = \sum_{\boldsymbol{k}} i(k_x \hat{u}_x + k_y \hat{u}_y + k_z \hat{u}_z) \exp(i\boldsymbol{k} \cdot \boldsymbol{x}) = 0 \tag{9.70}$$

which implies $\boldsymbol{k} \cdot \hat{\boldsymbol{u}}(\boldsymbol{k}, t) = 0$ for each wavevector $\boldsymbol{k}$.

9.3.4 The Navier Stokes equations in Fourier space

We can now use the Fourier expansion to recast the Navier Stokes equation in Fourier space. To do so we begin with

$$\frac{\partial u_i}{\partial t} + u_j \partial_j u_i = -\partial_i p + \nu \partial_{jj} u_i \tag{9.71}$$

(ignoring the possibility of a forcing term for now), using the convention of implicit summation over repeated indices. This becomes

$$\frac{\partial}{\partial t}\sum_{\boldsymbol{k}} \hat{u}_i(\boldsymbol{k},t)\exp(i\boldsymbol{k}\cdot\boldsymbol{x}) + \sum_{\boldsymbol{k}'} \hat{u}_j(\boldsymbol{k}',t)\exp(i\boldsymbol{k}'\cdot\boldsymbol{x})\partial_j \sum_{\boldsymbol{k}} \hat{u}_i(\boldsymbol{k},t)\exp(i\boldsymbol{k}\cdot\boldsymbol{x})$$
$$= -\partial_i \sum_{\boldsymbol{k}} \hat{p}(\boldsymbol{k},t)\exp(i\boldsymbol{k}\cdot\boldsymbol{x}) + \nu\partial_{jj}\sum_{\boldsymbol{k}} \hat{u}_i(\boldsymbol{k},t)\exp(i\boldsymbol{k}\cdot\boldsymbol{x}) \quad (9.72)$$

(note the difference between the subscripts i and the i that is $\sqrt{-1}$) and so

$$\sum_{\boldsymbol{k}} \frac{\partial \hat{u}_i}{\partial t}(\boldsymbol{k},t)\exp(i\boldsymbol{k}\cdot\boldsymbol{x}) + i\sum_{\boldsymbol{k},\boldsymbol{k}'} \hat{u}_j(\boldsymbol{k}',t)k_j\hat{u}_i(\boldsymbol{k},t)\exp(i(\boldsymbol{k}+\boldsymbol{k}')\cdot\boldsymbol{x})$$
$$= -i\sum_{\boldsymbol{k}} k_i\hat{p}(\boldsymbol{k},t)\exp(i\boldsymbol{k}\cdot\boldsymbol{x}) - \nu\sum_{\boldsymbol{k}} |\boldsymbol{k}|^2\hat{u}_i(\boldsymbol{k},t)\exp(i\boldsymbol{k}\cdot\boldsymbol{x}) \quad (9.73)$$

Multiplying by $\boldsymbol{k}''$ and integrating over the volume (using the orthogonality condition)

$$\frac{\partial \hat{u}_i}{\partial t}(\boldsymbol{k}'',t) + i\sum_{\boldsymbol{k}} k_j\hat{u}_j(\boldsymbol{k}''-\boldsymbol{k},t)\hat{u}_i(\boldsymbol{k},t) = -ik''_i\hat{p}(\boldsymbol{k}'',t) - \nu|\boldsymbol{k}''|^2\hat{u}_i(\boldsymbol{k}'',t) \quad (9.74)$$

or, renaming the $\boldsymbol{k}''$ as $\boldsymbol{k}$, and the $\boldsymbol{k}$ as $\boldsymbol{k}'$,

$$\frac{\partial \hat{u}_i}{\partial t}(\boldsymbol{k},t) + i\sum_{\boldsymbol{k}'} k'_j\hat{u}_j(\boldsymbol{k}-\boldsymbol{k}',t)\hat{u}_i(\boldsymbol{k}',t) = -ik_i\hat{p}(\boldsymbol{k},t) - \nu|\boldsymbol{k}|^2\hat{u}_i(\boldsymbol{k},t) \quad (9.75)$$

One advantage of the Fourier formalism is that it is very easy to solve for $\hat{p}$, using incompressibility. Indeed, dotting the momentum equation with $\boldsymbol{k}$, we have

$$\hat{p}(\boldsymbol{k},t) = -\sum_{\boldsymbol{k}'} \frac{k_l k'_j}{|\boldsymbol{k}|^2}\hat{u}_j(\boldsymbol{k}-\boldsymbol{k}',t)\hat{u}_l(\boldsymbol{k}',t) \quad (9.76)$$

(where we renamed the index i into l). As a result, we have

$$\frac{\partial \hat{u}_i}{\partial t}(\boldsymbol{k},t) = -i\sum_{\boldsymbol{k}'} k'_j\hat{u}_j(\boldsymbol{k}-\boldsymbol{k}',t)\left[\hat{u}_i(\boldsymbol{k}',t) - k_i\frac{k_l}{|\boldsymbol{k}|^2}\hat{u}_l(\boldsymbol{k}',t)\right] - \nu|\boldsymbol{k}|^2\hat{u}_i(\boldsymbol{k},t) \quad (9.77)$$

Finally, noting that

$$\boldsymbol{k}'\cdot\boldsymbol{u}(\boldsymbol{k}-\boldsymbol{k}') = -(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{u}(\boldsymbol{k}-\boldsymbol{k}') + \boldsymbol{k}\cdot\boldsymbol{u}(\boldsymbol{k}-\boldsymbol{k}') = -\boldsymbol{k}\cdot\boldsymbol{u}(\boldsymbol{k}-\boldsymbol{k}') \quad (9.78)$$

then

$$\frac{\partial \hat{u}_i}{\partial t}(\boldsymbol{k},t) = i\sum_{\boldsymbol{k}'} k_j\hat{u}_j(\boldsymbol{k}-\boldsymbol{k}',t)\left[\hat{u}_i(\boldsymbol{k}',t) - k_i\frac{k_l}{|\boldsymbol{k}|^2}\hat{u}_l(\boldsymbol{k}',t)\right] - \nu|\boldsymbol{k}|^2\hat{u}_i(\boldsymbol{k},t) \quad (9.79)$$

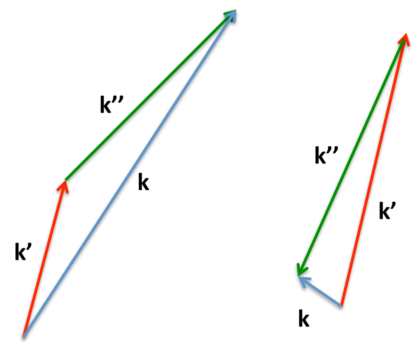

Figure 9.5: Left: two modes creating a higher wavenumber one; Right: two modes creating a lower wavenumber one.

which can also be rewritten as

$$\frac{\partial \hat{u}_i}{\partial t}(\boldsymbol{k},t) = i\sum_{\boldsymbol{k}'} k_j \left[\delta_{il} - \frac{k_i k_l}{|\boldsymbol{k}|^2}\right] \hat{u}_j(\boldsymbol{k}-\boldsymbol{k}',t)\hat{u}_l(\boldsymbol{k}',t) - \nu|\boldsymbol{k}|^2\hat{u}_i(\boldsymbol{k},t) \tag{9.80}$$

Note that the tensor

$$P_{il} = \delta_{il} - \frac{k_i k_l}{|\boldsymbol{k}|^2} \tag{9.81}$$

is a projection tensor onto the subspace normal to $\boldsymbol{k}$, therefore guaranteeing that the quantity in the sum is indeed orthogonal to $\boldsymbol{k}$, thus guaranteeing incompressibility for $\boldsymbol{u}$ at all points in time and space (this is indeed the only role of pressure in an incompressible fluid).

From this equation, we see that the evolution of a Fourier mode $\hat{\boldsymbol{u}}(\boldsymbol{k})$ is controlled by (1) viscous dissipation, which in Fourier space simply takes the form $-\nu|\boldsymbol{k}|^2\hat{\boldsymbol{u}}$, and (2) the nonlinear terms, which combine all other Fourier modes with wavectors $\boldsymbol{k}'$ and $\boldsymbol{k}''$ such that $\boldsymbol{k}'+\boldsymbol{k}'' = \boldsymbol{k}$. This is a clear expression of the energy cascade: modes with smaller wavenumbers can add up to transfer energy into a higher wavenumber one. But we also see that under the right circumstances, two high wavenumber modes of almost opposite directions can add up to feed energy into a low wavenumber mode (see Figure 9.5)

9.3.5 Energetics

We first note that by homogeneity, the kinetic energy density in the flow is constant and given by

$$E(t) = \frac{1}{2L^3}\int_0^L\int_0^L\int_0^L \boldsymbol{u}(\boldsymbol{x},t)\cdot\boldsymbol{u}(\boldsymbol{x},t)dxdydz = \frac{1}{2}\sum_{\boldsymbol{k}} \hat{\boldsymbol{u}}(\boldsymbol{k},t)\hat{\boldsymbol{u}}^*(\boldsymbol{k},t) \tag{9.82}$$

using the orthogonality property. As such, we can identify

$$\hat{E}(\boldsymbol{k},t) = \frac{1}{2}|\hat{\boldsymbol{u}}(\boldsymbol{k},t)|^2 \tag{9.83}$$

as the contribution from mode $\boldsymbol{k}$ to the kinetic energy density.

We can therefore construct an energy equation from the Navier Stokes equation in Fourier space, by using the product rule

$$\frac{\partial \hat{E}(\boldsymbol{k},t)}{\partial t} = \frac{1}{2}\left(\hat{\boldsymbol{u}}(\boldsymbol{k},t)\cdot\frac{\partial \hat{\boldsymbol{u}}^*}{\partial t}(\boldsymbol{k},t) + \hat{\boldsymbol{u}}^*(\boldsymbol{k},t)\cdot\frac{\partial \hat{\boldsymbol{u}}}{\partial t}(\boldsymbol{k},t)\right) \tag{9.84}$$

With a little work, this becomes

$$\begin{aligned}\frac{\partial \hat{E}(\boldsymbol{k},t)}{\partial t} \quad &= -\frac{i}{2}\sum_{\boldsymbol{k}'} k_j\left[\delta_{il} - \frac{k_i k_l}{|\boldsymbol{k}|^2}\right]\hat{u}_i(\boldsymbol{k},t)\hat{u}_j^*(\boldsymbol{k}-\boldsymbol{k}',t)\hat{u}_l^*(\boldsymbol{k}',t)\\ &+\frac{i}{2}\sum_{\boldsymbol{k}'} k_j\left[\delta_{il} - \frac{k_i k_l}{|\boldsymbol{k}|^2}\right]\hat{u}_i^*(\boldsymbol{k},t)\hat{u}_j(\boldsymbol{k}-\boldsymbol{k}',t)\hat{u}_l(\boldsymbol{k}',t)\\ &-2\nu|\boldsymbol{k}|^2\hat{E}(\boldsymbol{k},t)\\ &=\sum_{\boldsymbol{k}'} T(\boldsymbol{k},\boldsymbol{k}') - 2\nu|\boldsymbol{k}|^2\hat{E}(\boldsymbol{k},t)\end{aligned} \tag{9.85}$$

where the function $T(\boldsymbol{k},\boldsymbol{k}')$ represents the nonlinear transfer of energy from mode $\boldsymbol{k}'$ to mode $\boldsymbol{k}$ (which may be positive or negative). Summing this equation over all possible values of $\boldsymbol{k}$ must recover the evolution equation for $E(t)$, namely $dE/dt = -\epsilon$, which implies that

$$\epsilon(t) = \sum_{\boldsymbol{k}} 2\nu|\boldsymbol{k}|^2\hat{E}(\boldsymbol{k},t) \text{ and } \sum_{\boldsymbol{k},\boldsymbol{k}'} T(\boldsymbol{k},\boldsymbol{k}') = 0 \tag{9.86}$$

(Note that both of these equations can also be shown directly by performing the sum for T, and using the original definitions of ϵ in real space).

This shows that T is indeed a transfer function (i.e. a function that merely reshuffles energy between Fourier modes) and that the only actual energy loss is from the dissipation term. It also shows that dissipation is local in spectral space, and furthermore demonstrates that the decay rate is quadratic in the amplitude of the mode wavenumber $\boldsymbol{k}$, i.e. the decay rate of $\hat{E}(\boldsymbol{k})$ is $2\nu|\boldsymbol{k}|^2$. This confirms our intuition of the previous section that dissipation dominates at small scales (i.e. high wavenumbers).

9.3.6 The relationship between $\hat{E}(\boldsymbol{k})$ and $e(k)$

Previously, we defined the function $e(k)$ as the kinetic energy between modes of amplitude k and $k + dk$, such that

$$E^{tot} = V\int_0^\infty e(k,t)dk = L^3\int_0^\infty e(k,t)dk \tag{9.87}$$

This implies that the kinetic energy density is $E = \int_0^\infty e(k,t)dk$. In the limit that L tends to infinity, recall that we can expect the system to become more and more isotropic, which implies that $\hat{E}(\boldsymbol{k},t)$ becomes a function of $k = |\boldsymbol{k}|$ (and t) only. Let's rewrite

$$\begin{aligned} E &= \sum_{\boldsymbol{k}} \hat{E}(\boldsymbol{k},t) = \sum_{\boldsymbol{k}} \frac{\hat{E}(\boldsymbol{k},t)}{(2\pi/L)^3} dk_x dk_y dk_z \\ &= \frac{L^3}{(2\pi)^3} \int\int\int \hat{E}(k,t) k^2 \sin\theta_k dk d\theta_k d\phi_k = \frac{4\pi L^3}{(2\pi)^3} \int_0^\infty \hat{E}(k,t) k^2 dk \end{aligned} \tag{9.88}$$

after switching the integral to spherical coordinates in $\boldsymbol{k}$-space. This identifies

$$e(k,t) = \frac{L^3}{2\pi^2} \hat{E}(k,t) k^2 \tag{9.89}$$

(for sufficiently large L). We can verify that this is indeed dimensionally correct.

9.3.7 Relationship with 2-point correlation function

Recall our real-space definition of the two-point correlation function

$$\Phi_{ij}(\boldsymbol{r},t) = \langle u_i(\boldsymbol{x}+\boldsymbol{r},t) u_j(\boldsymbol{x},t)\rangle \tag{9.90}$$

where $\langle\rangle$ is a statistical average. Being statistically homogeneous, the quantity is equal to its volume average, so

$$\Phi_{ij}(\boldsymbol{r},t) = \langle\langle u_i(\boldsymbol{x}+\boldsymbol{r},t) u_j(\boldsymbol{x},t)\rangle_L\rangle \tag{9.91}$$

If we now substitute the Fourier expansions of $\boldsymbol{u}$, we obtain

$$\begin{aligned} \Phi_{ij}(\boldsymbol{r},t) &= \langle\langle \sum_{\boldsymbol{k}} \sum_{\boldsymbol{k}'} \hat{u}_i(\boldsymbol{k},t) \exp(i\boldsymbol{k}\cdot(\boldsymbol{x}+\boldsymbol{r})) \hat{u}_j(\boldsymbol{k}',t) \exp(i\boldsymbol{k}'\cdot\boldsymbol{x})\rangle_L\rangle \\ &= \langle \sum_{\boldsymbol{k}} \hat{u}_i(\boldsymbol{k},t)\hat{u}_j(-\boldsymbol{k},t)\exp(i\boldsymbol{k}\cdot\boldsymbol{r})\rangle & (9.92) \\ &= \langle \sum_{\boldsymbol{k}} \hat{u}_i(\boldsymbol{k},t)\hat{u}_j^*(\boldsymbol{k},t)\exp(i\boldsymbol{k}\cdot\boldsymbol{r})\rangle & (9.93) \end{aligned}$$

using orthogonality and the fact that $\boldsymbol{u}$ is real, which shows that the Fourier transform of $\Phi_{ij}(\boldsymbol{r},t)$ is

$$\hat{\Phi}_{ij}(\boldsymbol{k},t) = \langle \hat{u}_i(\boldsymbol{k},t)\hat{u}_j^*(\boldsymbol{k},t)\rangle \tag{9.94}$$

Since $\Phi_{ij}(\boldsymbol{r})$ is isotropic, so must $\hat{\Phi}_{ij}(\boldsymbol{k})$ also be, which implies (by the same arguments as put forward earlier) that

$$\hat{\Phi}_{ij}(\boldsymbol{k},t) = a(k,t)\delta_{ij} + b(k,t)k_i k_j \tag{9.95}$$

Incompressibility implies that

$$k_i\hat{\Phi}_{ij}(\boldsymbol{k},t) = \langle k_i\hat{u}_i(\boldsymbol{k},t)\hat{u}_j^*(\boldsymbol{k},t)\rangle = 0 \tag{9.96}$$

so

$$k_i \hat{\Phi}_{ij}(\boldsymbol{k},t) = a(k,t)k_j + b(k,t)|\boldsymbol{k}^2|k_j = 0 \to b(k,t) = -\frac{a(k,t)}{|\boldsymbol{k}^2|} \tag{9.97}$$

and therefore

$$\hat{\Phi}_{ij}(\boldsymbol{k},t) = a(k,t)\left(\delta_{ij} - \frac{k_i k_j}{|\boldsymbol{k}^2|}\right) \tag{9.98}$$

In addition, noting that

$$\langle \hat{E}(\boldsymbol{k},t)\rangle = \frac{1}{2}\langle \hat{u}_i(\boldsymbol{k},t)\hat{u}_i^*(\boldsymbol{k},t)\rangle = \frac{1}{2}\hat{\Phi}_{ii}(\boldsymbol{k},t) \tag{9.99}$$

we see that

$$\hat{\Phi}_{ii}(\boldsymbol{k},t) = 2a(k,t) \to \langle \hat{E}(\boldsymbol{k},t)\rangle = a(k,t) \tag{9.100}$$

so

$$\hat{\Phi}_{ij}(\boldsymbol{k},t) = \langle \hat{E}(\boldsymbol{k},t)\rangle \left(\delta_{ij} - \frac{k_i k_j}{|\boldsymbol{k}^2|}\right) \tag{9.101}$$

Finally, combining this with the definition of $e(k)$ that incorporates the isotropy of $\hat{E}$, we have

$$\hat{\Phi}_{ij}(\boldsymbol{k},t) = \frac{2\pi^2}{k^2 L^3}\langle e(k,t)\rangle \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right) \tag{9.102}$$

This shows, in spectral space, the duality between the two-point correlation function and the energy spectrum. As such, we see that the energy equation (9.85) in spectral space must be equivalent to the von Kármán-Howarth equation, although the actual relationship between the function $f(r,t)$ defined earlier and $e(k,t)$ remains to be clarified.

To do so, note that on the one hand

$$\frac{1}{2}\Phi_{ii}(\boldsymbol{r},t) = \frac{U^2(t)}{2r}\left[3\frac{\partial}{\partial r}(r^2 f) - r^2\frac{\partial f}{\partial r}\right] = \frac{U^2(t)}{2r^2}\frac{\partial}{\partial r}(r^3 f) \tag{9.103}$$

but this is also

$$\begin{aligned}
\frac{1}{2}\Phi_{ii}(\boldsymbol{r},t) &= \sum_{\boldsymbol{k}} \frac{1}{2}\hat{\Phi}_{ii}(\boldsymbol{k},t)\exp(i\boldsymbol{r}\cdot\boldsymbol{k}) \\
&= \sum_{\boldsymbol{k}} \frac{2\pi^2}{k^2 L^3}\langle e(k,t)\rangle \exp(i\boldsymbol{r}\cdot\boldsymbol{k}) \\
&= \left(\frac{L}{2\pi}\right)^3 \int\int\int \frac{2\pi^2}{k^2 L^3}\langle e(k,t)\rangle \exp(i\boldsymbol{r}\cdot\boldsymbol{k}) d^3\boldsymbol{k}
\end{aligned} \tag{9.104}$$

in the limit of L going to infinity. Since $\boldsymbol{r}$ is fixed in this problem, we can create a spherical coordinate system around that axis. As a result,

$$\begin{aligned}
\frac{1}{2}\Phi_{ii}(\boldsymbol{r},t) &= \left(\frac{1}{2\pi}\right)^3 \int_0^\infty \int_0^\pi \int_0^{2\pi} \frac{2\pi^2}{k^2} \langle e(k,t)\rangle \exp(irk\cos\theta_r) k^2 \sin\theta_r dk d\theta_r d\phi_r \\
&= \left(\frac{1}{2\pi}\right)^3 \int_0^\infty \int_0^\pi \frac{4\pi^3}{k^2} \langle e(k,t)\rangle \exp(irk\cos\theta_r) k^2 \sin\theta_r dk d\theta_r \\
&= \left(\frac{1}{2\pi}\right)^3 \int_0^\infty \int_{-1}^1 \frac{4\pi^3}{k^2} \langle e(k,t)\rangle \exp(irkv) k^2 dk dv \\
&= \frac{1}{2}\int_0^\infty \langle e(k,t)\rangle \frac{\exp(irk)-\exp(-irk)}{irk} dk \\
&= \int_0^\infty \langle e(k,t)\rangle \frac{\sin(rk)}{rk} dk \qquad (9.105)
\end{aligned}$$

Combining these two finally yields

$$\frac{U^2(t)}{2r^2}\frac{\partial}{\partial r}(r^3 f) = \int_0^\infty \langle e(k,t)\rangle \frac{\sin(rk)}{rk} dk \qquad (9.106)$$

Multiplying this by r^2, we can integrate both sides in r to relate f to e, as

$$\frac{U^2(t)}{2} r^3 f(r,t) = \int_0^\infty k^{-1}\langle e(k,t)\rangle \left[\int_0^r r' \sin(r'k) dr'\right] dk \qquad (9.107)$$

Since

$$\int_0^r r'\sin(r'k)dr' = \frac{\sin(kr) - rk\cos(kr)}{k^2} \qquad (9.108)$$

this becomes

$$f(r,t) = \frac{2}{U^2(r)}\int_0^\infty \langle e(k,t)\rangle \frac{\sin(kr) - rk\cos(kr)}{r^3k^3} dk \qquad (9.109)$$

Using this, a similar correspondance between the triple correlations in real space and the transfer function in spectral space, and a *lot* of algebra, we can finally recover (should we want to) the von Kárman-Howarth equation.

9.3.8 Integral scale and Taylor microscale

In Section 9.2.3, we defined the integral scale as

$$L_I = \int_0^\infty f(r,t)dr \qquad (9.110)$$

It is easy to verity that this is also

$$L_I = \frac{1}{2}\int_0^\infty \frac{1}{r^2}\frac{\partial}{\partial r}(r^3 f)dr \qquad (9.111)$$

so we write L_I is in terms of $e(k)$:

$$L_I = \frac{1}{U^2(t)} \int_0^\infty \int_0^\infty \langle e(k,t) \rangle \frac{\sin(rk)}{rk} dk dr \tag{9.112}$$

Now, the integral over r and k can be switched, and since $\int_0^\infty x^{-1} \sin(ax) dx = \pi/2$ if $a > 0$, then

$$L_I = \frac{\pi}{2U^2(t)} \int_0^\infty \frac{\langle e(k,t) \rangle}{k} dk = \frac{3\pi}{4\langle E \rangle} \int_0^\infty \frac{\langle e(k,t) \rangle}{k} dk \tag{9.113}$$

which reveals L_I to be proportional to the expectation value of k^{-1} over the kinetic energy spectrum (which therefore puts preferential weight on the more energetic eddies).

Similarly, we can revisit the definition of the Taylor microscale λ, which involves the second derivative of $f(r,t)$ near $r = 0$. We have

$$\frac{\partial^2 f}{\partial r^2} = \frac{2}{U^2(t)} \int_0^\infty k^{-3} \langle e(k,t) \rangle \frac{(12 - 5k^2 r^2) \sin(kr) + rk(k^2 r^2 - 12) \cos(kr)}{r^5} dk \tag{9.114}$$

Taylor expanding the integrand in the vicinity of $r = 0$ (by hand or using Wolfram alpha for instance), we finally get

$$\left. \frac{\partial^2 f}{\partial r^2} \right|_{r=0} = -\frac{2}{15U^2(t)} \int_0^\infty k^2 \langle e(k,t) \rangle dk \tag{9.115}$$

This is equal to $-\lambda^{-2}$, which shows that

$$\lambda = \left(\frac{2}{15U^2(t)} \int_0^\infty k^2 \langle e(k,t) \rangle dk \right)^{-1/2} \tag{9.116}$$

Now, recalling that we also have

$$\epsilon(t) = 2\nu \sum_{\boldsymbol{k}} |\boldsymbol{k}^2| \hat{E}(\boldsymbol{k},t) \tag{9.117}$$

we can again take the limit $L \to \infty$ to show that

$$\begin{aligned} \epsilon(t) \quad &= 2\nu \frac{L^2}{(2\pi)^3} \int \int \int |\boldsymbol{k}^2| \hat{E}(\boldsymbol{k},t) d^3\boldsymbol{k} = 2\nu \frac{L^3}{(2\pi)^3} \int_0^\infty \hat{E}(\boldsymbol{k},t) 4\pi k^4 dk \\ &= 2\nu \int_0^\infty e(k,t) k^2 dk \end{aligned}$$

As a result, we can write

$$\lambda = U(t) \sqrt{\frac{15\nu}{\epsilon(t)}} \tag{9.118}$$

which recovers exactly the definition we had in the previous section. However, with this approach, we can see that λ is directly related to the expectation value of k^2 over the kinetic energy spectrum, to the power of -1/2, i.e. the preferred dissipation lengthscale. This lengthscale is not η, as we discussed earlier, and now we can see why: even though $k = 1/\eta$ is larger, the kinetic energy associated with modes of wavenumbers close to η is very small. Instead, most of the dissipation occurs at a tradeoff wavenumber (which is close to $1/\lambda$) which is large enough for dissipation to be important, but also small enough for the kinetic energy of eddies at that wavenumber to be large.

Though this foray into the Fourier interpretation of turbulence, we have therefore demonstrated that it yields entirely equivalent results to those obtained from the statistical view of turbulence. This is not surprising, but provides confidence in the results derived, as well as means to interpret them in different ways depending on the lense used.