Chapter 9

Turbulence theory (non-rotating, non-stratified)

In the previous Chapters, we looked at the linear stability, weakly nonlinear behavior and energy stability properties of various fluid systems. What happens beyond the initial phases of instability, especially when the system is strongly unstable, was however not addressed, and it is now time to do so. It is generally the case that when a fluid is strongly unstable (i.e. far beyond the threshold for instability), the initial instability develops into *turbulence*. While we all have an intuitive notion of what turbulence is, mathematically it is somewhat hard to define. Nevertheless, we usually understand it to have two important properties: (1) it it nonlinear and chaotic in nature, and (2) it exhibits variability on a wide range of lengthscales and timescales. The chaotic nonlinear nature of turbulence makes it generally impossible to analyze analytically in an exact manner, so the vast majority of studies to date have been numerical or experimental. And unfortunately, the wide range of timescales and lengthscales involved in a turbulent fluid imply that this problem is numerically very challenging, and significant progress on that front has only really been possible in the past 2 decades. Because of the formidable complexity of the problem, turbulence is generally agreed to be the "most important unsolved problem of classical physics" (Feynmann).

In this Chapter, I will describe a few aspects of turbulence. This is by no means intended to be comprehensive! Instead, I am to provide background material that will allow the reader to read and understand some of the more complex recent forays into the problem. I will limit the discussion, for simplicity, to the case of fluid flows that are neither rotating, nor stratified.

Sections 1 and 2 are based on the textbooks by Davidson (Turbulence) and Pope (Turbulent flows).

9.1 Phenomenology

9.1.1 A vast range of scales

One of the few properties of turbulence that most scientists will agree to is that it contains a *vast range of scales*. A good example of what a turbulent flow may be is shown in the figure below, which is of the eruption from Mount Pinatubo in 1991. The picture shows very clearly that there are dynamics on every scale from the *outer scale* (i.e. the size of the entire plume), down to the smallest scale the eye can see. (Of course, in that picture, the turbulence is both stratified, probably influenced by the effects of the Earth's rotation, and by the presence of ash particles – but it is still a good visual example of the kind of turbulent dynamics we care about in this Chapter).

Figure 9.1: Photo of the Mount Pinatubo eruption of 1991; from Wikipedia.

The reason why turbulent fluid flows have such a wide range of scales can easily be understood by considering a 1D equation that is *almost* the Navier-Stokes equation but not quite:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2} \tag{9.1}$$

This is the *viscous Burger's equation.* Let's consider the following initial condition: $u(x,0) = \sin(x)$, in the unbounded domain. Then, we see that at a time $t = \Delta t$ later, where Δt is small,

$$u(x,\Delta t) \simeq u(x,0) + \Delta t\left[-u\frac{\partial u}{\partial x} + \nu\frac{\partial^2 u}{\partial x^2}\right]_{t=0} \simeq (1-\nu\Delta t)\sin(x) - \Delta t\frac{\sin(2x)}{2} \tag{9.2}$$

and at a timestep Δt after that,

$$
\begin{aligned}
u(x, 2\Delta t) \quad & \simeq u(x, \Delta t) + \Delta t \left[-u \frac{\partial u}{\partial x} + \nu \frac{\partial^2 u}{\partial x^2} \right]_{t=\Delta t} \qquad (9.3) \\
& \simeq \left[(1 - \nu \Delta t) \sin(x) - \Delta t \frac{\sin(2x)}{2} \right] - \nu \Delta t \left[(1 - \nu \Delta t) \sin(x) - 2\Delta t \sin(2x) \right] \\
& -\Delta t \left[(1 - \nu \Delta t) \sin(x) - \Delta t \frac{\sin(2x)}{2} \right] \left[(1 - \nu \Delta t) \cos(x) - \Delta t \cos(2x) \right]
\end{aligned}
$$

which will create terms in $\sin(3x)$ and $\sin(4x)$, and so forth. The nonlinear terms in this equation clearly create structure on smaller and smaller scales as time goes by. Of course in this case, solving the Burger's equation exactly reveals that the small scale features correspond to the development of a sawtooth profile (rather than something that would look like a 1D velocity profile of, say, the Mount Pinatubo eruption). But the general idea that the nonlinearities in the momentum equation cause energy to transfer from the large scales to the small scales nevertheless holds.

9.1.2 The energy cascade

Let's now consider the full incompressible Navier-Stokes equations,

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\rho_m} \nabla p + \nu \nabla^2 \boldsymbol{u} \quad (+\boldsymbol{F}) \qquad (9.4)$$

together with $\nabla \cdot \boldsymbol{u} = 0$, to describe the evolution of a turbulent flow. Without loss of generality, in all that follows we take $\rho_m = 1$ (by appropriately choosing the units of mass). In this flow, the properties of the turbulence can either be evolving with time after being triggered by some initial conditions, or, can be maintained in a statistically stationary state by the presence of an external force $\boldsymbol{F}$.

We now consider the energetics of the flow. The total kinetic energy equation derived from (9.4) in the usual way (i.e. dotting the momentum equation with $\boldsymbol{u}$) is

$$\frac{\partial}{\partial t} \left(\frac{u_i u_i}{2} \right) + u_j \partial_j \left(\frac{u_i u_i}{2} \right) = -u_i \partial_i p + \nu u_i \partial_{jj} u_i \quad (+u_i F_i) \qquad (9.5)$$

using Einstein's convention of summation over repeated indices. Then, using the fact that $\partial_i u_i = 0$, we have

$$\frac{\partial}{\partial t} \left(\frac{u_i u_i}{2} \right) + \partial_j \left(u_j \frac{u_i u_i}{2} \right) = -\partial_i (u_i p) + \nu \partial_j (u_i \partial_j u_i) - \nu (\partial_j u_i)(\partial_j u_i) \quad (+u_i F_i) \qquad (9.6)$$

which can finally be written as

$$\frac{\partial E}{\partial t} + \partial_j (u_j E + u_j p - \nu \partial_j E) = -\nu (\partial_j u_i)(\partial_j u_i) \quad (+u_i F_i) \qquad (9.7)$$

where $E = \sum_i u_i^2/2$ is the kinetic energy density (kinetic energy per unit volume, recalling that $\rho_m = 1$). Wherever possible, relevant terms were written in conservative form (i.e. as the divergence of a flux).

This expression clearly shows that the only non-conservative terms (on the right-hand-side) are the local kinetic energy production rate from the body force (namely $\boldsymbol{u} \cdot \boldsymbol{F}$) if the latter is present, and the local kinetic energy dissipation rate

$$\epsilon = \nu(\partial_j u_i)(\partial_j u_i) = \nu|\nabla \boldsymbol{u}|^2 \tag{9.8}$$

All the other terms represent energy fluxes (diffusive or advective), which merely move energy around in space, and disappear when integrated over a sufficiently large volume with appropriate boundary conditions. From an energetic point of view, we therefore see that energy is input on the large scales (assuming the body force $\boldsymbol{F}$, or the initial conditions, are relatively large scale), but is dissipated mostly on the small scales, because the energy dissipation rate $\nu|\nabla \boldsymbol{u}|^2$ depends on velocity gradients (which are largest on the smaller scales, see later for more on this topic).

Combining the ideas of nonlinear production of small scales, together with the fact that energy is produced on large scales and dissipated on small scales, Richardson introduced the notion of a *turbulent energy cascade*. He assumed that the turbulence can be viewed as being made of *eddies*. On the largest scales, eddies have size L and velocity U (associated with the forcing or initial conditions) and do not feel the viscosity. In other words, their Reynolds number $Re_L = UL/\nu$, which we recall from previous lectures is the ratio of the nonlinear terms to the viscous terms, is much smaller than 1. The eddies become unstable, nonlinearly giving rise to smaller scale eddies (see Figure 9.2) to whom they transfer energy. This transfer continues until the eddies are so small that their own Reynolds number Re_l (based on their small scale l and velocity $u(l)$) is of order unity, and viscous effects finally become important and stabilize them.

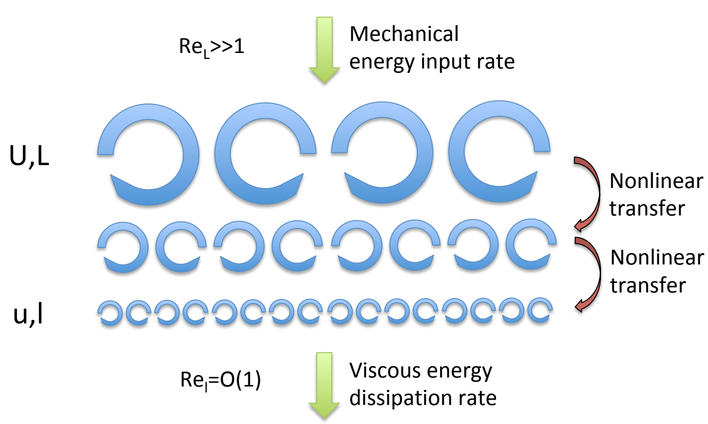

Figure 9.2: Richardon's idea of energy cascade

From an energetic point of view, and assuming the turbulence is in a statistically stationary state, this idea implies that the energy that is mechanically introduced in a fluid on the large scales (often called the *injection scale*), must then be transferred through the turbulent cascade without any loss, until the viscous scale is reached and energy can finally be dissipated viscously. As such, the rate of kinetic energy input into the system must be equal to the transfer rate from large scales to small scales, which must in turn be equal to the viscous dissipation rate ϵ, and this quantity therefore uniquely characterizes the kinetic energy transfer rate through the system. Note that in a statistically stationary state, ϵ is constant, but in a run-down experiment, ϵ is a function of time.

9.1.3 Kolmogorov's hypotheses

Enters Kolmogorov, who went on to propose several hypotheses to model the turbulence. Following Richardson, he assumed that there are three distinct ranges of scales in (non-rotating, non-stratified) turbulence:

- The outer scales, i.e. any scale that is sufficiently large to know about the boundary conditions applied to the fluid, or the overall shape of the system.
- The viscous scales, i.e. any scale that is sufficiently small to be affected by viscosity
- The inertial scales, i.e. any scale in between that is both sufficiently small to be unaware of the global system scales, yet large enough to be unaffected by viscosity.

In addition, he assumed that far from the outer scale (and in the absence of rotation or stratification) the turbulence should be homogenous and isotropic, implying (among other things) that a single lengthscale is sufficient to characterize an eddy.

Turbulent eddies in the inertial range, being unaware of either viscosity or the outer scale, can only know about their own lengthscale l and their own velocity scale $u(l)$. From a dimensional perspective, the only way to create a kinetic energy transfer rate (whose dimension is velocity squared over time) from these two quantities is:

$$\epsilon \propto \frac{u^3}{l} \tag{9.9}$$

showing that the velocity of an eddy of scale l has to be

$$u(l) \propto (l\epsilon)^{1/3} \tag{9.10}$$

From that, we can also construct the turnover time of an eddy of scale l as

$$\tau(l) = \frac{l}{u(l)} \propto \left(\frac{l^2}{\epsilon}\right)^{1/3} \tag{9.11}$$

We therefore see that both u and τ decrease with the scale l (so the eddies are slower, but turn over faster)

While not exactly applicable there, this formula must nevertheless smoothly match onto the properties of the outer scales, so we also expect that as $l \to L$,

$$\epsilon \simeq \frac{U^3}{L} \tag{9.12}$$

As such, we then also have

$$u(l) \simeq U \left(\frac{l}{L}\right)^{1/3} \tag{9.13}$$

$$\tau(l) = \frac{L}{U}\left(\frac{l}{L}\right)^{2/3} \tag{9.14}$$

and the Reynolds number based on the scale l is then

$$Re_l = \frac{u(l)l}{\nu} = Re_L \left(\frac{l}{L}\right)^{4/3} \tag{9.15}$$

At the other end of the cascade, the flow begins to know about viscosity. As such, it is now aware of *two* dimensional quantities, namely ϵ and ν. With two dimensional quantities, it is now possible to construct a unique characteristic lengthscale, velocity scale and timescale using dimensional analysis. Indeed, since ϵ has the dimension of a velocity squared over time (or equivalently, length squared over time cubed), and since ν has dimension of length squared over time, we can create these as

$$\begin{aligned} \tau_\eta &= \left(\frac{\nu}{\epsilon}\right)^{1/2} \text{ is the Kolmogorov timescale} \\ \eta &= \left(\frac{\nu^3}{\epsilon}\right)^{1/4} \text{ is the Kolmogorov lengthscale} \\ u_\eta &= (\epsilon\nu)^{1/4} \text{ is the Kolmogorov velocity scale} \end{aligned} \tag{9.16}$$

Note that although these definitions were created purely from a dimensional argument, we have

$$Re_\eta = \frac{u_\eta \eta}{\nu} = 1 \tag{9.17}$$

showing that η is indeed the scale at which viscous effects are important. We therefore also have

$$Re_L \left(\frac{\eta}{L}\right)^{4/3} = 1 \to \frac{\eta}{L} = Re_L^{-3/4} \tag{9.18}$$

and so

$$\frac{u_\eta}{U} = \left(\frac{\eta}{L}\right)^{1/3} = Re_L^{-1/4} \tag{9.19}$$

$$\frac{\tau_\eta}{L/U} = \left(\frac{l}{L}\right)^{2/3} = Re_L^{-1/2} \tag{9.20}$$

which all characterize the ratios of the small (length-, velocity-, time-) scales in the system to their corresponding outer scales. It is clear from these scalings that the larger the Reynolds number Re_L, the wider the separation between the outer scales and the viscous scales.

9.1.4 The turbulent energy spectrum

While based on very simple dimensional arguments, this first glimpse into Kolmogorov's theory of turbulence (which goes far beyond this) already provides enormous insight into many observations of real turbulent flows, in particular when it comes to the so-called *turbulent energy spectrum*. It is quite common to consider the amount of energy between wavenumber k and wavenumber $k + dk$, written as $e(k)dk$. With this definition, the total kinetic energy in a homogeneous isotropic system is simply

$$E^{tot} = V \int_0^\infty e(k)dk \tag{9.21}$$

(again, recalling that $\rho_m = 1$), where V is the total volume considered. From a dimensional perspective, $e(k)$ has the dimension of a velocity squared divided by a wavenumber (so $e(k)dk$ has the dimension of a velocity squared). In Kolmogorov's theory, we found that an eddy of size l has a velocity $u(l) \propto (\epsilon l)^{1/3}$ so a kinetic energy $\propto (\epsilon l)^{2/3} \propto \epsilon^{2/3} k^{-2/3}$. Hence, dimensionally speaking, we must have $e(k) \propto \epsilon^{2/3} k^{-2/3}/k = \epsilon^{2/3} k^{-5/3}$. This is the famous $k^{-5/3}$ law of turbulence. It is appropriate for isotropic turbulence in the inertial range in between the outer and viscous scales, which are $1/L$ and $1/\eta$ in wavenumber space. As a result, we expect an energy spectrum $e(k)$ that looks like the one in Figure 9.3a.

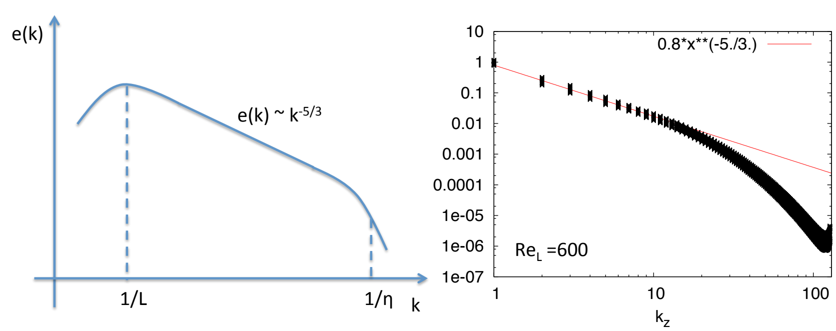

Figure 9.3: Left: Idealized Kolmogorov spectrum, between injection wavenumber $1/L$ and Kolmogorov wavenumber $1/\eta$. Right: Actual spectrum of a $Re_L = 600$ unstratified shear flow triply-periodic DNS. Note that $\eta/L = 1/120$ for this Reynolds number. k_z here is in units of L^{-1}.

For comparison, Figure 9.3b shows the actual energy spectrum as a function of vertical wavenumber k_z of a 3D Direct Numerical Simulation of an unstratified shear flow, forced on the largest scale ($L = 2\pi$), with $Re_L = 600$. We clearly see a well-defined cascade with the $k^{-5/3}$ law, for k_z between 1 and 10. With this outer Reynolds number, we expect $\eta = L/120$, so the Kolmogorov wavenumber in this simulation (and with this non-dimensionalization) should be about 120. We therefore see that the energy density begins deviating away from the $k^{-5/3}$ law substantially before the Kolmogorov scale. More on this later.

9.2 Statistical theory of turbulence

Given that most turbulence studies in the past were done using laboratory or in-situ experiments, it was not possible to measure the full velocity field at every point in time and space. Instead, it was common to measure the fluid velocity as a function of time, at well-defined positions in the fluid, using fixed velocity probes. As such, an important focus of many early studies of turbulence theory was concerned with statistical properties of these time series, and in particular, the correlation functions of the fluid flow, which are relatively easy to measure. In this section, we will study what predictions Kolmorogov's assumptions of homogeneous isotropic turbulence with a well-defined inertial range imply for these statistical properties of the turbulence.

9.2.1 Definitions

Let's assume we are able to perform the same experiment over and over again, and each time measuring flow quantities at various points in the flow, as a function of time. We can construct statistical averages of the measurements, such as for instance the rms value of the various components of the velocity $\boldsymbol{u}$ at a point $\boldsymbol{x}$, and time t, as

$$U_{x,rms}(\boldsymbol{x}, t) = \langle u_x^2(\boldsymbol{x}, t)\rangle^{1/2}, \tag{9.22}$$

$$U_{y,rms}(\boldsymbol{x}, t) = \langle u_y^2(\boldsymbol{x}, t)\rangle^{1/2}, \tag{9.23}$$

$$U_{z,rms}(\boldsymbol{x}, t) = \langle u_z^2(\boldsymbol{x}, t)\rangle^{1/2}, \tag{9.24}$$

where the angular bracket represents an average over many different realizations of the same experiment (note here that we have assumed for simplicity that there is no mean flow so $\langle \boldsymbol{u}(\boldsymbol{x}, t)\rangle = 0$). Without further information on the turbulence, these quantities will be functions of position and time.

We can also define the two-point correlation function

$$\Phi_{ij}(\boldsymbol{x}, \boldsymbol{r}, t) = \langle u_i(\boldsymbol{x} + \boldsymbol{r}, t)u_j(\boldsymbol{x}, t)\rangle \tag{9.25}$$

where the indices i and j represent the various directions (x,y or z). This quantity, as its name suggests, measures how correlated u_i and u_j are a distance $\boldsymbol{r}$ away from one another. Without any further information on the turbulence, this tensor will be a function of position $\boldsymbol{x}$, of the separation between the two points $\boldsymbol{r}$, and time.

9.2.2 Homogeneous isotropic turbulence

As we now demonstrate, Kolmorogov's assumptions of homogeneous isotropic turbulence strongly constraints the statistical properties of the turbulence. Indeed if the turbulence is homogenous, all of its statistics must be independent of *where* the measurements are taken. As a result, the rms velocities and the tensor Φ_{ij} cannot depend on position $\boldsymbol{x}$, so $U_{x,rms}(\boldsymbol{x},t) = U_{x,rms}(t)$, and similarly for the other components, and $\Phi_{ij}(\boldsymbol{x},\boldsymbol{r},t) = \Phi_{ij}(\boldsymbol{r},t)$. Interestingly, we see that when $\boldsymbol{r} = \boldsymbol{0}$ then

$$\sum_i \Phi_{ii}(\boldsymbol{0},t) = \langle u_x^2 + u_y^2 + u_z^2 \rangle = U_{x,rms}^2(t) + U_{y,rms}^2(t) + U_{z,rms}^2(t) = 2\langle E(t)\rangle \tag{9.26}$$

is twice the statistically averaged kinetic energy density (i,e, kinetic energy per unit volume) of the fluid. That quantity is independent of position, by the assumption of homogeneity.

If the turbulence is isotropic as well (which is assumed here), then by definition it has no distinct direction. This implies that $\langle u_x^2\rangle = \langle u_y^2\rangle = \langle u_z^2\rangle$, and we will therefore define

$$U(t) = \langle u_x^2\rangle^{1/2} = \sqrt{\langle \frac{2}{3}E(t)\rangle} \tag{9.27}$$

as the 1D rms velocity (i.e. the rms velocity of a single component of $\boldsymbol{u}$). The function $\Phi_{ij}(\boldsymbol{r},t)$ must also be isotropic (i.e. invariant with respect to any rotation of the coordinate system); as such, it can only depend on the magnitude of $\boldsymbol{r}$ and not its direction. It has been shown that the only possibility for creating an isotropic tensor that is function of a vector $\boldsymbol{r}$ is as

$$\Phi_{ij}(\boldsymbol{r},t) = A(r,t)\delta_{ij} + B(r,t)r_i r_j \tag{9.28}$$

Without loss of generality (since the system is isotropic) we can take $\boldsymbol{r}$ to be (for instance) in the x direction. Then, the only non-zero entries of Φ are the diagonal terms, which are

$$\begin{aligned} \Phi_{xx}(r\mathbf{e}_x,t) &= A(r,t) + B(r,t)r^2 \\ \Phi_{yy}(r\mathbf{e}_x,t) = \Phi_{zz}(r\mathbf{e}_x,t) &= A(r,t) \end{aligned} \tag{9.29}$$

The quantity $\Phi_{xx}(\boldsymbol{r},t)$ is proportional to the normalized *longitudinal* autocorrelation function, which is the autocorrelation of the velocity field $\boldsymbol{u}$ projected along the direction $\boldsymbol{r}$, usually called $f(r,t)$ and generally defined as

$$f(r,t) = \frac{\langle [\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r},t)\cdot \boldsymbol{e}_r]\,[\boldsymbol{u}(\boldsymbol{x},t)\cdot \boldsymbol{e}_r]\rangle}{\langle [\boldsymbol{u}(\boldsymbol{x},t)\cdot \boldsymbol{e}_r]^2\rangle} \tag{9.30}$$

Because of isotropy, we can take $\boldsymbol{r}$ to be in any direction, so let's choose $\boldsymbol{r} = r\boldsymbol{e}_x$. In that case, the denominator is just $U^2(t)$, while the numerator is just $\Phi_{xx}(r\boldsymbol{e}_x,t)$. This then implies $\Phi_{xx}(r\mathbf{e}_x,t) = A(r,t) + B(r,t)r^2 = f(r,t)U(t)^2$.

The quantity $\Phi_{yy}(\mathbf{r},t)$ is similarly proportional to the normalized *transverse* autocorrelation function of the velocity field $\boldsymbol{u}$ (i.e. the autocorrelation function of flow in the direction perpendicular to $\boldsymbol{r}$), usually called $g(r,t)$, as

$$\Phi_{yy}(r\mathbf{e}_x,t) = A(r,t) = g(r,t)U^2(t) \tag{9.31}$$

Using these identities, we can rewrite Φ_{ij} in terms of the functions $U(t)$, $f(r,t)$ and $g(r,t)$ as

$$\Phi_{ij}(\boldsymbol{r},t) = U^2(t)\left[g(r,t)\delta_{ij} + (f(r,t)-g(r,t))\frac{r_i r_j}{r^2}\right] \tag{9.32}$$

So far, we have only used symmetries of the problem. We can also use continuity, i.e. $\nabla\cdot\boldsymbol{u}=0$ to further constrain the form of Φ_{ij}. Indeed, we have that

$$\sum_i \frac{\partial}{\partial r_i}\langle u_i(\boldsymbol{x}+\boldsymbol{r},t)u_j(\boldsymbol{x},t)\rangle = \sum_i \langle\frac{\partial}{\partial r_i}u_i(\boldsymbol{x}+\boldsymbol{r},t)u_j(\boldsymbol{x},t)\rangle = 0 \tag{9.33}$$

which implies that

$$\sum_i \frac{\partial}{\partial r_i}\Phi_{ij}(\boldsymbol{r},t) = \sum_i U^2(t)\left[\frac{\partial g}{\partial r}\frac{\partial r}{\partial r_i}\delta_{ij} + \frac{\partial}{\partial r}(f(r,t)-g(r,t))\frac{r_i r_j}{r^2}\frac{\partial r}{\partial r_i}\right.$$
$$\left.-2(f(r,t)-g(r,t))\frac{r_i r_j}{r^3}\frac{\partial r}{\partial r_i} + (f(r,t)-g(r,t))\frac{r_j}{r^2} + (f(r,t)-g(r,t))\frac{r_i}{r^2}\delta_{ij}\right] = 0 \tag{9.34}$$

Since $\partial r/\partial r_i = r_i/r$, and $\sum_i r_i^2 = r^2$, we can simplify this to

$$\frac{\partial g}{\partial r}\frac{r_j}{r} + \frac{\partial}{\partial r}(f(r,t)-g(r,t))\frac{r_j}{r} - 2(f(r,t)-g(r,t))\frac{r_j}{r^2} + 4(f(r,t)-g(r,t))\frac{r_j}{r^2} = 0 \tag{9.35}$$

This then yields an equation for $g(r,t)$:

$$\frac{\partial}{\partial r}f(r,t) + 2\frac{f(r,t)-g(r,t)}{r} = 0 \tag{9.36}$$

or in other words,

$$g(r,t) = f(r,t) + \frac{r}{2}\frac{\partial}{\partial r}f(r,t) = \frac{1}{2r}\frac{\partial}{\partial r}(r^2 f) \tag{9.37}$$

so finally,

$$\Phi_{ij}(\boldsymbol{r},t) = \frac{U^2(t)}{2r}\left[\frac{\partial}{\partial r}(r^2 f)\delta_{ij} - \frac{\partial}{\partial r}f(r,t)r_i r_j\right] \tag{9.38}$$

This relationship is very important because it shows that all of the 2-point correlation information in homogeneous isotropic incompressible turbulence can be captured by knowing a single function, namely $f(r,t)$. This does not tell us what that function is, but instead, states that this function is the only one needed to characterize this information.

Finally, combining the fact that f is a nondimensional function – and should therefore be written as a function of non-dimensional quantities only – with Kolmogorov's assumption that in the inertial range or below, the system can only know about either ν or ϵ, we deduce that the function f must be of the form

$$f(r,t) = \hat{f}\left(\frac{r}{\eta}, \frac{t}{\tau_\eta}\right) = \hat{f}(\hat{r}, \hat{t}) \tag{9.39}$$

as long as $r \ll L$. This is indeed found experimentally.

9.2.3 The integral scale and the Taylor microscale

From the function $f(r,t)$, which is dimensionless by construction, we can construct at least two important lengthscales associated with the flow. The first one is called the integral scale, and it is a good measure of the lengthscale of energy bearing eddies at any point in time:

$$L_I(t) = \int_0^\infty f(r,t)dr \tag{9.40}$$

(it is easy to verify that this indeed has the dimensions of a length, since dr is a length). A nice physical interpretation of this lengthscale will emerge when looking at Fourier representations of the flow, see later.

A second important lengthscale is the Taylor[1] microscale, which is derived from the Taylor[2] expansion of $f(r)$. First, noting that $f(0,t) = 1$ by construction, and that $f(r,t) = f(-r,t)$ because of isotropy, we have

$$f(r,t) = 1 - \frac{r^2}{2\lambda^2} + ... \tag{9.41}$$

where

$$\lambda^{-2} = -\left.\frac{\partial^2 f}{\partial r^2}\right|_{r=0} \tag{9.42}$$

Dimensionally speaking, λ is a lengthscale. Going back to the relationship

[1] After G.I. Taylor
[2] After B. Taylor

between f and $\Phi_{xx}(r\boldsymbol{e}_x, t)$, namely $\Phi_{xx}(r\mathbf{e}_x, t) = f(r,t)U(t)^2$, we also see that

$$\begin{aligned}
\left.\frac{\partial^2 f}{\partial r^2}\right|_{r=0} &= U(t)^{-2} \lim_{r\to 0} \frac{\partial^2}{\partial r^2} \Phi_{xx}(r\mathbf{e}_x, t) \\
&= U(t)^{-2} \lim_{r\to 0} \langle \frac{\partial^2}{\partial r^2} u_x(\boldsymbol{x} + r\boldsymbol{e}_x, t) u_x(\boldsymbol{x}, t)\rangle \\
&= U(t)^{-2} \lim_{r\to 0} \langle \left.\frac{\partial^2 u_x}{\partial x^2}\right|_{\boldsymbol{x}+r\boldsymbol{e}_x,t} u_x(\boldsymbol{x}, t)\rangle \\
&= U(t)^{-2} \langle \left.\frac{\partial^2 u_x}{\partial x^2}\right|_{\boldsymbol{x},t} u_x(\boldsymbol{x}, t)\rangle \\
&= U(t)^{-2} \langle \frac{\partial}{\partial x}\left(u_x \frac{\partial u_x}{\partial x}\right) - \left(\frac{\partial u_x}{\partial x}\right)^2 \rangle \\
&= U(t)^{-2} \frac{\partial}{\partial x} \langle u_x \frac{\partial u_x}{\partial x}\rangle - \langle \left(\frac{\partial u_x}{\partial x}\right)^2 \rangle \\
&= -U(t)^{-2} \langle \left(\frac{\partial u_x}{\partial x}\right)^2 \rangle
\end{aligned} \tag{9.43}$$

where we have used homogeneity to note that spatial derivatives of any flow statistic should be 0. Finally, it can be shown that the kinetic energy dissipation ϵ introduced in the previous section is directly related to $\langle (\partial u_x/\partial x)^2 \rangle$ in homogeneous isotropic turbulence (see for instance exercise 5.28 page 133 in the Pope textbook), as

$$\epsilon(t) = 15\nu \langle \left(\frac{\partial u_x}{\partial x}\right)^2 \rangle \tag{9.44}$$

As a result, the Taylor microscale λ introduced earlier can be expressed only as a function of the dissipation $\epsilon(t)$ and the rms 1D velocity $U(t)$ as

$$\lambda(t) = U(t)\sqrt{\frac{15\nu}{\epsilon(t)}} \tag{9.45}$$

An important consequence of this relationship is that it can help measure ϵ using only pointwise velocity measurements at nearby positions in the fluid (to compute the longitudinal two-point correlation function $f(r,t)$, and then its curvature at $r = 0$ to measure λ).

Note that the Taylor microscale, despite knowing about both ν and ϵ, is *not* the same as the Kolmogorov scale. To see this,

$$\frac{\lambda}{\eta} = U\sqrt{\frac{15\nu}{\epsilon}} \frac{\epsilon^{1/4}}{\nu^{3/4}} = \sqrt{15}U(\epsilon\nu)^{-1/4} = \sqrt{15}\frac{U}{u_\eta} = \sqrt{15}Re_L^{1/4} \tag{9.46}$$

As such, for large enough Re_L, $\lambda \gg \eta$. For the simulation presented in Figure 9.3b, for instance, $\lambda = \sqrt{15}(600^{1/4})\eta \simeq 20\eta$. We see that this corresponds to a wavenumber that is close to the place where the spectrum begins to deviate away from the Kolmogorov law. As such, it is often said that the inertial range ends around λ^{-1} (rather than around η^{-1}), and that the quantity of relevance to determine whether the flow has a turbulent cascade or not is

$$\begin{aligned} Re_\lambda &= \frac{u(\lambda)\lambda}{\nu} = \frac{u(\lambda)}{U}\frac{\lambda}{L}Re_L = \left(\frac{\lambda}{L}\right)^{4/3} Re_L \\ &= \left(\frac{\lambda}{\eta}\right)^{4/3} = 15^{2/3} Re_L^{1/3} \end{aligned} \tag{9.47}$$

If $Re_\lambda \gg 1$, then a substantial cascade can indeed be present.

9.2.4 A statistical view of the Navier-Stokes equations

So far, all of the properties we have derived concerning the tensor Φ_{ij} were made purely based on arguments of symmetry and incompressibility, without ever making use of the fact that the flow field $\boldsymbol{u}$ actually has to satisfy the governing equation (9.4). Using the Navier-Stokes equations, however, it is possible to derive a constraint on the evolution of the function $f(r,t)$, as demonstrated by von Kármán & Howarth (1938). The full derivation is quite complicated, and I encourage you to look at the original paper for detail. The general lines, however, go as follows. Let's focus on the case of decaying turbulence, ie. $\boldsymbol{F} = 0$, for simplicity.

First, note that

$$\frac{\partial}{\partial t}\Phi_{ij}(\boldsymbol{r},t) = \langle u_i(\boldsymbol{x},t)\frac{\partial}{\partial t}u_j(\boldsymbol{x}+\boldsymbol{r},t)\rangle + \langle u_j(\boldsymbol{x}+\boldsymbol{r},t)\frac{\partial}{\partial t}u_i(\boldsymbol{x},t)\rangle \tag{9.48}$$

Using the Navier-Stokes equations and incompressibility, as well as a large number of algebraic manipulations this becomes

$$\frac{\partial \Phi_{ij}}{\partial t} = \frac{\partial}{\partial r_k}(S_{ikj} + S_{jki}) + 2\nu\nabla_r^2\Phi_{ij} \tag{9.49}$$

where ∇_r is the gradient with respect to $\boldsymbol{r} = (r_x, r_y, r_z)$, and where we have introduced the three-point correlation tensor

$$S_{ijk}(\boldsymbol{r},t) = \langle u_i(\boldsymbol{x},t)u_j(\boldsymbol{x},t)u_k(\boldsymbol{x}+\boldsymbol{r},t)\rangle \tag{9.50}$$

which, by assumptions of homogeneity, is independent of $\boldsymbol{x}$.

Arguments of homogeneity and isotropy similar to the ones given for the tensor Φ_{ij} earlier imply that it is possible to write S_{ijk} in terms of a single function $K(r,t)$ only, namely

$$K(r,t) = \frac{S_{xxx}(r\boldsymbol{e}_x,t)}{U^3(t)} \tag{9.51}$$

such that

$$S_{ijk}(\boldsymbol{r},t) = U\left[\frac{K - rK'}{2r^3}r_ir_jr_k + \frac{2K+rK'}{4r}(r_i\delta_{jk} + r_j\delta_{ik}) - \frac{K}{2r}r_k\delta_{ij}\right] \tag{9.52}$$

where $K' = \partial K/\partial r$. Substituting this into the evolution equation for Φ_{ij} (and using the equation relating Φ_{ij} to $f(r,t)$), we then obtain the famous von Kármán-Howarth equation for homogeneous isotropic turbulence:

$$\frac{\partial}{\partial t}\left[U^2(t)f(r,t)\right] = \frac{U^3(t)}{r^4}\frac{\partial}{\partial r}[r^4 K(r,t)] + 2\nu\frac{U^2(t)}{r^4}\frac{\partial}{\partial r}\left(r^4\frac{\partial f}{\partial r}\right) \tag{9.53}$$

This is complemented by the evolution equation for $U(t)$, that can be derived for instance by taking the statistical average of the energy equation (9.7):

$$\frac{\partial}{\partial t}\langle E\rangle + \langle\partial_j(u_j E + u_j p - \nu\partial_j E)\rangle = -\langle\nu(\partial_j u_i)(\partial_j u_i)\rangle \tag{9.54}$$

Noting that the derivative and the statistical average commute, and that the spatial gradients of the statistical averages must be 0, we simply have

$$\frac{\partial U^2}{\partial t} = -\frac{2}{3}\epsilon(t) \tag{9.55}$$

There is, however, no evolution equation for $K(r,t)$ at this order. We therefore see that while the von Kármán-Howarth equation is exact, it suffers from a *closure* problem – $K(r,t)$ must be known to evolve $f(r,t)$, but it isn't. Attempts to create an evolution equation for $K(r,t)$ would only result in the appearance of a 4th order tensor, and a corresponding 4-th order correlation function. We will return to the closure problem later in this Chapter.

Looking at the von Kármán-Howarth equation, we see that there are two terms, which have clear interpretations: the term containing ν arises from the viscous stress term, and therefore describes the effect of viscosity on the evolution of f. The term containing K, on the other hand, comes from the nonlinear terms in the momentum equation and must therefore describe all inertial processes (including the cascade of energy to smaller scales).

In the limit where inertial terms are negligible (i.e. for instance, in the very final phases of the turbulent decay of energy), then the equation becomes

$$\frac{\partial}{\partial t}\left[U^2(t)f(r,t)\right] \simeq 2\nu\frac{U^2(t)}{r^4}\frac{\partial}{\partial r}\left(r^4\frac{\partial f}{\partial r}\right) \tag{9.56}$$

so, using (9.55)

$$U^2\frac{\partial f}{\partial t} - \frac{2}{3}\epsilon f \simeq 2\nu\frac{U^2(t)}{r^4}\frac{\partial}{\partial r}\left(r^4\frac{\partial f}{\partial r}\right) \tag{9.57}$$

where, from the discussion of the Taylor microscale, we had

$$\epsilon = -15\nu U^2 \left.\frac{\partial^2 f}{\partial r^2}\right|_{r=0} \tag{9.58}$$

Substituting one into the other, we see that U^2 disappears, leaving

$$\frac{\partial f}{\partial t} \simeq 2\nu\frac{1}{r^4}\frac{\partial}{\partial r}\left(r^4\frac{\partial f}{\partial r}\right) - 10 f\nu \left.\frac{\partial^2 f}{\partial r^2}\right|_{r=0} \tag{9.59}$$

It's easy to show (by substitution) that there exists an exact solution to this equation, namely

$$\lim_{t\to\infty} f(r,t) = \exp\left(-\frac{r^2}{8\nu t}\right) \tag{9.60}$$

We note that this expression satisfies (9.39), as required.

We then have

$$\lambda = \left(\left. \frac{\partial^2 f}{\partial r^2} \right|_{r=0} \right)^{-1/2} = 2\sqrt{\nu t} \tag{9.61}$$

from which we deduce

$$\epsilon(t) = \frac{15}{4}\frac{U^2(t)}{t} \tag{9.62}$$

Substituting this into the evolution equation for U^2, we finally get

$$\frac{dU^2}{dt} = -\frac{2}{3}\epsilon(t) = -\frac{5}{2}\frac{U^2(t)}{t} \rightarrow U^2(t) \propto t^{-5/2} \tag{9.63}$$

In other words, in the final stages of decay of turbulence, we expect the total kinetic energy to decay as $t^{-5/2}$.

Finally, it is important to note that various other properties of the statistics of the flow can be derived using similar lines of argument, the most important of which is the well-known Kolmogorov four-fifth law for the three point correlations, which states that for $\eta \ll r \ll L$ (i.e. for r within the inertial range), then

$$\langle |u_i(\boldsymbol{x}+\boldsymbol{r},t) - u_i(\boldsymbol{x},t)|^3 \rangle = -\frac{4}{5}\epsilon r \tag{9.64}$$

(where i can be any direction, since the flow is isotropic), see textbooks for detail.

Many of these laws have been successfully tested in DNS of homogeneous isotropic turbulence at high Reynolds number, as well as in laboratory experiments, for which a sufficiently wide inertial range exists.