6.5 Linear theory for stratified shear flows (inviscid case)

In the previous sections, we only looked at unstratified shear flows, i.e, flows of constant density ρ_0, where gravity plays no role in the momentum equation. We now include the effects of stratification by considering a background stratified fluid of density $\bar{\rho}(z)$, in hydrostatic equilibrium so that

$$\frac{\partial \bar{p}}{\partial z} = -\bar{\rho}(z)g \qquad (6.67)$$

We shall further assume that the fluid is *stably* stratified, i.e. that $\bar{\rho}(z)$ decreases with height, so that the system is not convectively unstable. Stratified shear flows are extremely common in geophysics and astrophysics. For instance, parts of the Earth's atmosphere are stably stratified, and are subject to winds – a classical example of a stratified shear flow. Similarly, many regions of the ocean are stably stratified and subject to currents.

We saw earlier that in the absence of stratification, shear instabilities are energetically possible (as determined from energy stability arguments) as long as the Reynolds number is large-enough, but the details of what makes a flow linearly stable or unstable are quite complex. Let us now see what happens to shear instabilities when a stable stratification is present.

6.5.1 Energetics of stratified shear flows

As in the previous lecture, we can first look at the energetics of stratified shear instabilities to determine the conditions under which they are, at least approximately, favorable or unfavorable. Let's consider two parcels of fluid in a shear flow $\bar{u}(z) = \bar{u}(z)e_x$, stratified with the background density profile $\bar{\rho}(z)$, shown in Figure 6.6. The lower parcel is at $z = 0$ and the upper parcel is at $z = \epsilon$. We ask the question again of whether a mixing event increases or lowers the total energy of the system. If the total energy increases, this means that energy has to be provided for the mixing event to proceed, which is therefore an unfavorable configuration for the development of instabilities. On the other hand if the total energy is lowered in the mixing event, the shear instabilities are energetically favorable.

In the situation considered here, the mixing event homogenizes both the density and the momentum of the two parcels[2]. Hence, if the two parcels have densities $\bar{\rho}(0)$ and $\bar{\rho}(\epsilon)$ respectively, then after the event their common density is ρ_m where

$$\rho_m = \frac{1}{2}(\bar{\rho}(0) + \bar{\rho}(\epsilon)) \qquad (6.68)$$

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[2]You may wonder *why* it is the momentum, rather than the velocities that are homogenized. That's because, at the heart of fluid dynamics, are the two conservation equations for mass (or density) and momentum. Mixing events are merely events that stir the fluid while conserving these quantities.
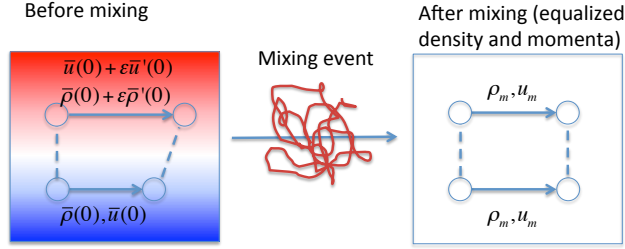
Figure 6.6: Energetics of stratified shear flows. A mixing event homogenizes the density of the two parcels, as well as their momenta

Similarly, if the two parcels initially have momenta $\bar{\rho}(0)\bar{u}(0)$ and $\bar{\rho}(\epsilon)\bar{u}(\epsilon)$, after the event they both have the same momentum $\rho_m u_m$, where

$$\rho_m u_m = \frac{1}{2}(\bar{\rho}(0)\bar{u}(0) + \bar{\rho}(\epsilon)\bar{u}(\epsilon)) \tag{6.69}$$

which defines what u_m is.

Let's now calculate the total energy of the system, which now has contributions both from the kinetic energy $\frac{1}{2}\bar{\rho}\bar{u}^2$, and from the potential energy $\bar{\rho}gz$. Before the mixing event, the energy is

$$E_i = \frac{1}{2}\left(\bar{\rho}(0)\bar{u}(0)^2 + \bar{\rho}(\epsilon)\bar{u}(\epsilon)^2\right) + g\epsilon\bar{\rho}(\epsilon) \tag{6.70}$$

while after the event

$$E_f = \rho_m u_m^2 + \rho_m g\epsilon \tag{6.71}$$

Using successive Taylor expansions for small ϵ, and lots of algebra, one can then show that

$$\Delta E = E_i - E_f \simeq \frac{\epsilon^2}{2}\left(\frac{1}{2}\bar{\rho}(0)\bar{u}'(0)^2 + g\bar{\rho}'(0)\right) \tag{6.72}$$

This expression can be interpreted in two equivalent ways. We saw that in the absence of stratification, shear instabilities are always energetically favorable. We now see that they remain energetically favorable in the presence of an unstable density gradient $\rho'(0) > 0$ (which is really not surprising), but can be stabilized by a sufficiently strong negative (i.e. stable) density gradient. Alternatively, (6.72) can also be interpreted to mean that instabilities on a stably stratified flow can be energetically favorable provided the shear $\bar{u}'(0)$ is sufficiently strong.

6.5.2   The Richardson number for stratified shear flows

The results of the previous section suggest that whether stratified shear flows are stable or unstable depends on the non-dimensional ratio

$$\frac{g\bar{\rho}'}{\rho\bar{u}'^2} = \frac{N^2}{S^2} \equiv J(z) \tag{6.73}$$

recalling that N^2 can also be written as $N^2 = g\bar{\rho}'/\rho$. This number is called the *Richardson number* (after Richardson). Because J is based on local gradients, it is often called the *gradient Richardson number*.

The Richardson number, as we saw, is related to the ratio of the potential energy lost while moving density around, to the kinetic energy gained from the shear. For instability to occur, J should be *smaller* than a factor of order unity, while stability is likely when J is larger than unity. It is important to realize, however, that these are just approximate arguments based on energetic considerations. They do not replace a thorough linear stability analysis, nor do they provide any rigorous results on energy stability. However, they give us a good idea of what to expect next!

6.5.3 Global stability analysis of inviscid, non-diffusive stratified shear flows

Having briefly studied the stability of unstratified fluids, we now go back to look at the case of stratified flows, ignoring viscosity and thermal dissipation for simplicity. We consider fluids that have a background density profile $\bar{\rho}(z) = \rho_{0z}z$, and a background shear flow $\bar{u}(z)$. We use the Boussinesq approximation in which the density perturbations are related to the temperature perturbations as $\tilde{\rho}/\rho_0 = -\alpha\tilde{T}$, and the background temperature profile $\bar{T}$ is similarly related to $\bar{\rho}$ so that $\bar{T}(z) = T_{0z}z$.

Following similar steps as we did for the global analysis of Section 6.3 we find that perturbations satisfy

$$\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{w}}{\partial z} = 0$$

$$\frac{\partial \tilde{u}}{\partial t} + \bar{u}(z)\frac{\partial \tilde{u}}{\partial x} + \tilde{w}\frac{d\bar{u}}{dz} = -\frac{\partial \tilde{p}}{\partial x}$$

$$\frac{\partial \tilde{w}}{\partial t} + \bar{u}(z)\frac{\partial \tilde{w}}{\partial x} = -\frac{\partial \tilde{p}}{\partial z} + \alpha g\tilde{T}$$

$$\frac{\partial \tilde{T}}{\partial t} + \bar{u}(z)\frac{\partial \tilde{T}}{\partial x} + \tilde{w}T_{0z} = 0 \tag{6.74}$$

Letting $\tilde{q} = \hat{q}\exp(ik_x x + \lambda t)$, we get

$$ik_x\hat{u} + \frac{d\hat{w}}{dz} = 0$$

$$\lambda\hat{u} + ik_x\bar{u}(z)\hat{u} + \hat{w}\frac{d\bar{u}}{dz} = -ik_x\hat{p}$$

$$\lambda\hat{w} + ik_x\bar{u}(z)\hat{w} = -\frac{d\hat{p}}{dz} + \alpha g\hat{T}$$

$$\lambda\hat{T} + ik_x\bar{u}(z)\hat{T} + \hat{w}T_{0z} = 0 \tag{6.75}$$

The usual steps (eliminating $\hat{p}$, then $\hat{u}$), yield a coupled system for $\hat{w}$ and $\hat{T}$:

$$(\lambda + ik_x\bar{u}(z))\left(\frac{d^2\hat{w}}{dz^2} - k_x^2\hat{w}\right) - ik_x\hat{w}\frac{d^2\bar{u}}{dz^2} = -k_x^2\alpha g\hat{T}$$

$$(\lambda + ik_x\bar{u}(z))\hat{T} + \hat{w}T_{0z} = 0 \tag{6.76}$$

so that

$$(\lambda + ik_x\bar{u}(z))^2\left(\frac{d^2\hat{w}}{dz^2} - k_x^2\hat{w}\right) - ik_x(\lambda + ik_x\bar{u}(z))\hat{w}\frac{d^2\bar{u}}{dz^2} = N^2k_x^2\hat{w} \tag{6.77}$$

where $N^2 = \alpha gT_{0z}$. If $\lambda = -ik_xc$, then we obtain the *Taylor-Goldstein* equation:

$$(\bar{u}(z) - c)\left(\frac{d^2\hat{w}}{dz^2} - k_x^2\hat{w}\right) - \hat{w}\frac{d^2\bar{u}}{dz^2} + \frac{N^2}{\bar{u}(z) - c}\hat{w} = 0 \tag{6.78}$$

This equation reduces to the Rayleigh equation when there is no stratification ($N = 0$).

As usual, there are unstable modes provided there exist solutions with $\Im(c) > 0$. As in the case of inviscid shear flows, we can easily show that if a solution $\hat{w}$ exists with eigenvalue c, then $\hat{w}^*$ is also a solution with eigenvalue c^* – which means, as before, that there are either neutral modes ($\Im(c) = 0$), or complex conjugate pairs of modes with one of them being unstable. The actual eigenmodes and their eigenvalues, however, must usually be found numerically.

Interestingly, a sufficient condition for *stability*, i.e. for the non-existence of modes with $\Im(c) > 0$ is due to Miles and Howard. They proved that a stratified shear flow is *stable* provided

$$N^2(z) > \frac{1}{4}\left(\frac{d\bar{u}}{dz}\right)^2 \tag{6.79}$$

everywhere in the fluid. This can be rewritten as: A necessary condition for *instability* is that

$$J(z) < \frac{1}{4} \text{ somewhere in the fluid} \tag{6.80}$$

where $J(z)$ is the gradient Richardson number defined in equation 6.73. This criterion is often called the *Richardson criterion for instability*. Note that $J(z) < \frac{1}{4}$ somewhere in the fluid does not guarantee instability – i.e. it is not a sufficient condition for instability, merely a necessary one. On the other hand, most flows are usually indeed destabilized as soon as the minimum value of J in the domain considered drops below a certain value, which is usually not very far below 1/4.

The general proof of this theorem is due to Howard. Starting from the Taylor-Goldstein equation, Howard suggested the following change of variable:

$$H = \frac{\hat{w}}{\sqrt{\bar{u}(z) - c}} \tag{6.81}$$

With this new variable, the equation can be cast in the self-adjoint form

$$\frac{d}{dz}\left[(\bar{u}(z) - c)\frac{dH}{dz}\right] - \left[k_x^2(\bar{u}(z) - c) + \frac{1}{2}\bar{u}'' + \frac{\frac{1}{4}(\bar{u}')^2 - N^2}{\bar{u}(z) - c}\right]H = 0 \quad (6.82)$$

While this appears to make the equation look more, rather than less, complicated, this self-adjoint form is *very* useful. Indeed, let us multiply the equation by the complex conjugate of H, and integrate it with respect to z.

$$\int_{z_1}^{z_2} H^* \frac{d}{dz}\left[(\bar{u}(z) - c)\frac{dH}{dz}\right] - \left[k_x^2(\bar{u}(z) - c) + \frac{1}{2}\bar{u}'' + \frac{\frac{1}{4}(\bar{u}')^2 - N^2}{\bar{u}(z) - c}\right]|H|^2 = 0$$
$$(6.83)$$

We now integrate the first term by parts. With standard boundary conditions where $\hat{w}$ vanishes on the boundaries, the boundary terms are zero, and we are left with

$$\int_{z_1}^{z_2} (\bar{u}(z) - c)\left|\frac{dH}{dz}\right|^2 + \left[k_x^2(\bar{u}(z) - c) + \frac{1}{2}\bar{u}'' + \frac{\frac{1}{4}(\bar{u}')^2 - N^2}{\bar{u}(z) - c}\right]|H|^2 = 0 \quad (6.84)$$

The imaginary part of this equation is

$$c_I \int_{z_1}^{z_2} \left|\frac{dH}{dz}\right|^2 + k_x^2|H|^2 = c_I \int_{z_1}^{z_2} \frac{\frac{1}{4}(\bar{u}')^2 - N^2}{|\bar{u}(z) - c|^2}|H|^2 \quad (6.85)$$

where c_I is the imaginary part of c. In order to have instability, we have to have a non-zero c_I. Since the LHS is strictly positive, this means that the RHS must also be strictly positive. A necessary condition for this to happen is that the quantity

$$\frac{1}{4}(\bar{u}')^2 - N^2 > 0 \quad (6.86)$$

somewhere in the domain. This is equivalent to (6.80) and therefore proves the theorem.

It is therefore interesting, but not too surprising, to see that the general necessary condition for linear instability (namely $J < 1/4$) is pretty similar to the hand-waving energy argument put forward in the previous section!

6.5.4 Example: the stratified Bickley jet.

Let's go back to the case of the Bickley jet and see what the effect of stratification on the instability is. The real and imaginary components of the Taylor-Goldstein

equation are:

$$(\bar{u}(z) - c_R)\left(\frac{d^2\hat{w}_R}{dz^2} - k_x^2\hat{w}_R\right) + c_I\left(\frac{d^2\hat{w}_I}{dz^2} - k_x^2\hat{w}_I\right)$$

$$-\hat{w}_R\bar{u}''(z) + N^2\frac{(\bar{u}(z) - c_R)\hat{w}_R - c_I\hat{w}_I}{|\bar{u}(z) - c|^2} = 0$$

$$(\bar{u}(z) - c_R)\left(\frac{d^2\hat{w}_I}{dz^2} - k_x^2\hat{w}_I\right) - c_I\left(\frac{d^2\hat{w}_R}{dz^2} - k_x^2\hat{w}_R\right)$$

$$-\hat{w}_I\bar{u}''(z) + N^2\frac{(\bar{u}(z) - c_R)\hat{w}_I + c_I\hat{w}_R}{|\bar{u}(z) - c|^2} = 0 \qquad (6.87)$$

Figure 6.7 shows the solution c_I of this eigenvalue problem, for the varicose mode, for increasing values of the stratification as measured by N^2. We see that as N^2 increases, c_I (and therefore the growth rate as well, since the latter is equal to k_x times c_I) decreases. This is a very typical result that is true in *almost* all situations, and is consistent with our basic physical interpretation of the instability: the larger N^2 is, the stronger the background density gradient, and the higher the potential energy cost of the perturbations to the flow. For large enough N^2, we expect the instability to be stabilized altogether[3].
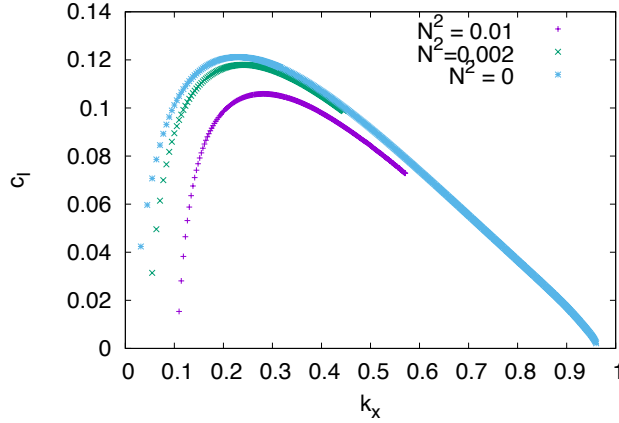

Figure 6.7: Imaginary part of c for the varicose mode of the Bickley jet for increasing values of N^2.

6.5.5 The effect of viscosity and diffusion on stratified shear instabilities

In what we have done so far, both viscosity and the effect of diffusion (of temperature, or of whatever scalar is responsible for the density stratification) were neglected.

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[3]Figure to be updated with larger values of N^2

Generally speaking, including viscosity has a stabilizing effect on the instability, either reducing its growth rate, or suppressing it entirely depending on the value of the Reynolds number. In fact, it can be shown with some effort that the energy stability criterion for unstratified shear flows is the *same* as the one for stratified shear flows, showing that energy stability is possible provided the Reynolds number is small enough.

The effect of thermal diffusion, on the other hand, goes the *opposite* way: diffusive stratified shear flows are generally more unstable than their non-diffusive counterparts. To understand why this is the case, note that diffusion has a tendency to smear out any density perturbations, so that any parcel of fluid slowly moved up into a less dense region will, over time, also become less dense, and any parcel slowly moved down into a denser region will, over time, become denser. As a result, the potential energy cost of any vertical fluid motion is less than the one estimated from assuming that the parcels retain their original densities while moving. This has the important consequence of relaxing the Richardson criterion, allowing instabiliy to occur even when $J(z)$ is everywhere greater than 1/4. Unfortunately, no formal theorem similar to the Miles-Howard theorem exists in this case, but this is a rather ubiquitous empirical finding of most attempts at solving the diffusive extension of the Taylor-Goldstein equation.

Diffusive shear instabilities are relevant in the Earth's atmosphere, and perhaps also in stellar interiors. They are probably not important in the ocean, however.