6.4 The viscous theory for shear instabilities

6.4.1 The background flow and the importance of viscosity

In the previous Section, we studied inviscid shear flows. These turn out to be somewhat peculiar in the sense that any profile $\bar{u}(z)$ could be used for the background shear. In reality, however, shear flows usually arise from a balance between forcing and viscous dissipation, and there is a single background solution for a given forcing and a given set of boundary conditions. Indeed, if we try to solve

$$\rho_m \left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} \right) = -\nabla p + \rho_m \nu \nabla^2 \boldsymbol{u} + F(z) \boldsymbol{e}_x \tag{6.45}$$

(where we have arbitrarily chosen to take the force as acting in the $x-$direction), then the only steady-state solution (assuming, say, periodic boundary conditions in x) is such that

$$\rho_m \nu \nabla^2 \bar{\boldsymbol{u}} + F(z) \boldsymbol{e}_x = 0 \tag{6.46}$$

or in other words,

$$\bar{\boldsymbol{u}} = \bar{u}(z) \boldsymbol{e}_x \tag{6.47}$$

where $\bar{u}(z)$ satisfies

$$\frac{d^2 \bar{u}}{dz^2} = -\frac{F(z)}{\rho_m \nu} \tag{6.48}$$

The actual solution $\bar{u}(z)$ will then depend on what is assumed in terms of the boundary conditions in z. For a constant force F_0, for instance, with no-slip boundaries at $z = 0$ and $z = 1$ (so $\bar{u}(0) = \bar{u}(1) = 0$, we find that the solution is

$$\bar{u}(z) = -\frac{F_0}{\rho_m \nu} \frac{z(z-1)}{2} \tag{6.49}$$

or, in other words, a parabolic profile (called a *Poiseuille flow*). Other forces and other boundary conditions will similarly yield other background flow profiles $\bar{u}(z)$.

Since viscosity is key in selecting the background flow, it is often not a good idea to neglect it in the perturbation equations. For this reason, we now proceed to analyze the stability of shear flows in the presence of viscosity.

6.4.2 Linear stability

As in the case of inviscid shear flows, we now let $\boldsymbol{u} = \bar{\boldsymbol{u}} + \tilde{\boldsymbol{u}}$, and substitute this into the momentum equation. We get

$$\frac{\partial \tilde{\boldsymbol{u}}}{\partial t} + \bar{\boldsymbol{u}} \cdot \nabla \tilde{\boldsymbol{u}} + \tilde{\boldsymbol{u}} \cdot \nabla \bar{\boldsymbol{u}} = -\frac{1}{\rho_m} \nabla \tilde{p} + \nu \nabla^2 \tilde{\boldsymbol{u}} \tag{6.50}$$

With the same steps as in the case of inviscid flows, we arrive at

$$(\lambda+ik_x\bar{u}(z))\left(\frac{d^2\hat{w}}{dz^2}-k_x^2\hat{w}\right)-ik_x\hat{w}\frac{d^2\bar{u}}{dz^2}=\nu\left(\frac{d^2}{dz^2}-k_x^2\right)\left(\frac{d^2\hat{w}}{dz^2}-k_x^2\hat{w}\right) \quad (6.51)$$

which can then be transformed into the *Orr-Sommerfeld* equation:

$$(\bar{u}(z)-c)D\hat{w}-\hat{w}\frac{d^2\bar{u}}{dz^2}=-i\frac{\nu}{k_x}D^2\hat{w} \quad (6.52)$$

using, as before $\lambda = -ik_x c$ and where the operator $D \equiv d^2/dz^2 - k_x^2$

By contrast with Rayleigh's equation, the Orr-Sommerfeld equation is always regular (for $\nu \neq 0$), since the coefficient in front of the highest derivative is never 0. It is therefore much easier to find solutions numerically. However, the equation itself is of higher order and very rarely has any analytical solution. Some theorems associated with properties of solutions of the Orr-Sommerfeld equation are discussed by Drazin & Reid in the textbook *Hydrodynamic Stability*. The most important set of results concerning the stability of viscous shear flows are summarized in Chapter 4 (where they use the notation $R \propto 1/\nu$ for the Reynolds number, and $\alpha = k_x$). We see that

- In general, viscosity has a tendency to stabilize shear flows for very large values of ν (small values of R). For instance, the range of unstable modes for the Bickley jet (e.g. case (d)) is null below a critical value of R, and then gradually increases to recover the inviscid range for large R.
- This last statement is in fact true of all cases: for $R \to \infty$, the inviscid limit is indeed recovered (so it is not a singular limit of the equations).
- Interestingly, however, we also find that linear shear flows (which are linearly stable for all wavenumbers in the inviscid limit), can be unstable for an intermediate range of values of the viscosity. This is a peculiar case where viscosity can have a destabilizing effect on a system.

6.4.3 Energy stability for viscous linear shear flows

To finish this section on the stability of unstratified shear flows, we now look again at the problem of energy stability, using the method discussed in the context of convection. Let's consider a domain of height L_z, and horizontal size L_x, and assume for the moment that there is a linear background shear flow $\bar{u}(z) = S\boldsymbol{e}_z$. We assume that all the perturbations to that background are periodic in L_x and L_z. We now look at the energetics of perturbations $\tilde{\boldsymbol{u}}$ around that state. The governing equations are :

$$\begin{aligned}
&\frac{\partial\tilde{u}}{\partial x}+\frac{\partial\tilde{w}}{\partial z}=0\\
&\frac{\partial\tilde{u}}{\partial t}+\tilde{\boldsymbol{u}}\cdot\nabla\tilde{u}+Sz\frac{\partial\tilde{u}}{\partial x}+S\tilde{w}=-\frac{1}{\rho_0}\frac{\partial\tilde{p}}{\partial x}+\nu\nabla^2\tilde{u}\\
&\frac{\partial\tilde{w}}{\partial t}+\tilde{\boldsymbol{u}}\cdot\nabla\tilde{w}+Sz\frac{\partial\tilde{w}}{\partial x}=-\frac{1}{\rho_0}\frac{\partial\tilde{p}}{\partial z}+\nu\nabla^2\tilde{w}
\end{aligned} \quad (6.53)$$

Non-dimensionalizing the distances with respect to the vertical size L_z of the domain, and the velocity in terms of SL_z, we get the non-dimensional equations

$$\begin{aligned}
&\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \\
&\frac{\partial u}{\partial t} + \boldsymbol{u} \cdot \nabla u + z\frac{\partial u}{\partial x} + w = -\frac{\partial \tilde{p}}{\partial x} + \frac{1}{\mathrm{Re}}\nabla^2 u \\
&\frac{\partial w}{\partial t} + \boldsymbol{u} \cdot \nabla w + z\frac{\partial w}{\partial x} = -\frac{\partial \tilde{p}}{\partial z} + \frac{1}{\mathrm{Re}}\nabla^2 w
\end{aligned} \tag{6.54}$$

where everything is now implicitely non-dimensional variables and where

$$\mathrm{Re} = \frac{SL_z^2}{\nu} \tag{6.55}$$

is the Reynolds number of the flow. The Reynolds number is another very famous number in fluid dynamics that measures the ratio of the inertial terms ($\boldsymbol{u} \cdot \nabla \boldsymbol{u}$) to the viscous terms ($\nu\nabla^2\boldsymbol{u}$). The larger the Reynolds number is, the less important viscosity is. In the limit of very large Reynolds number, viscosity should be negligible.

Using the usual trick of dotting the momentum equation with $\boldsymbol{u}$, we get the very simple energy equation

$$\frac{\partial E}{\partial t} = -\langle uw \rangle - \frac{1}{\mathrm{Re}}\langle |\nabla \boldsymbol{u}|^2 \rangle \equiv \mathcal{H}(\boldsymbol{u}) \tag{6.56}$$

As in the case of convection, we then try to determine when energy stability occurs, ie. when $\mathcal{H}(\boldsymbol{u})$ is negative for all possible divergence-free velocity fields.

We first maximize $\mathcal{H}(\boldsymbol{u})$ under the constraints that $\nabla \cdot \boldsymbol{u} = 0$ and the dissipation functional $\mathcal{D} = D_0$. To do so, we create the functional

$$\mathcal{S} = -\langle uw \rangle + \Lambda_1 \langle \frac{1}{\mathrm{Re}}|\nabla \boldsymbol{u}|^2 - D_0 \rangle + \langle \Lambda_2(x,z)\nabla \cdot \boldsymbol{u} \rangle \tag{6.57}$$

with the two Lagrange multipliers Λ_1 and $\Lambda_2(x,z)$. This defines the Lagrangian

$$\mathcal{L} = -uw + \Lambda_1(\frac{1}{\mathrm{Re}}|\nabla \boldsymbol{u}|^2 - D_0) + \Lambda_2(x,z)\nabla \cdot \boldsymbol{u} \tag{6.58}$$

The Euler-Lagrange equations for this maximization process are:

$$\begin{aligned}
-w &= \frac{\partial \Lambda_2}{\partial x} + 2\frac{\Lambda_1}{\mathrm{Re}}\nabla^2 u \\
-u &= \frac{\partial \Lambda_2}{\partial z} + 2\frac{\Lambda_1}{\mathrm{Re}}\nabla^2 w
\end{aligned} \tag{6.59}$$

together with the two constraints. Eliminating Λ_2 between the two equations, we get

$$\frac{\partial u}{\partial x} - \frac{\partial w}{\partial z} = 2\frac{\Lambda_1}{\mathrm{Re}}\nabla^2\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \tag{6.60}$$

Since we are working in 2D we can use a streamfunction such that $u = \partial\phi/\partial z$ and $w = -\partial\phi/\partial x$. The equation above becomes

$$2\frac{\partial^2\phi}{\partial x\partial z} = 2\frac{\Lambda_1}{\mathrm{Re}}\nabla^4\phi \tag{6.61}$$

Assuming solutions of the kind $\phi(x,z) \sim \exp(ik_x x + ik_z z)$, this yields

$$\Lambda_1 = -\mathrm{Re}\frac{k_x k_z}{(k_x^2+k_z^2)^2} \tag{6.62}$$

Let's now go back to the original energy equation, and calculate its right-hand-side:

$$\begin{aligned}\frac{\partial E}{\partial t} &= -\langle uw\rangle - \frac{1}{\mathrm{Re}}\langle\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\rangle \\ &= \langle\frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial z}\rangle - \frac{1}{\mathrm{Re}}\langle\left(\frac{\partial^2\phi}{\partial x\partial z}\right)^2 + \left(\frac{\partial^2\phi}{\partial z^2}\right)^2 + \left(\frac{\partial^2\phi}{\partial x^2}\right)^2 + \left(\frac{\partial^2\phi}{\partial z\partial x}\right)^2\rangle \\ &= -k_x k_z\langle|\phi|^2\rangle - \frac{(k_x^2+k_z^2)^2}{\mathrm{Re}}\langle|\phi|^2\rangle \\ &= -\frac{(k_x^2+k_z^2)^2}{\mathrm{Re}}(1-\Lambda_1)\langle|\phi|^2\rangle\end{aligned} \tag{6.63}$$

This implies that for the system to be energy-stable ($dE/dt < 0$), we simply need $\Lambda_1 < 1$ where $k_x^2 + k_y^2$ is not allowed to be identically 0 (otherwise $dE/dt = 0$). Since we can rewrite Λ_1 as

$$\Lambda_1 = -\mathrm{Re}\frac{\cos\theta\sin\theta}{k^2} = -\frac{\mathrm{Re}}{2}\frac{\sin 2\theta}{k^2} \tag{6.64}$$

where $k^2 = k_x^2 + k_z^2$ and $\cos\theta = k_x/k$, then we see that Λ_1 is maximum for angles $\theta = -\pi/4$ and $\theta - 3\pi/4$, in which case $\sin(2\theta) = -1$, but continuously decreases with increasing k^2. Hence, the maximum value of Λ_1 is for $k_x = \pm k_z$, and for the smallest non-zero available value of k that lies at these angles. This implies

$$\max\Lambda_1 = \frac{\mathrm{Re}}{2}\max_{k_x=\pm k_y}\frac{1}{k_x^2+k_z^2} = \frac{\mathrm{Re}}{4}\frac{1}{\min(k_x^2,k_z^2)} = \mathrm{Re}\max(\hat{L}_x^2, 1) \tag{6.65}$$

where $\hat{L}_x = L_x/L_z$ is the horizontal length of the domain in units of L_z. To get to the last expression we have used the fact that the minimum wavenumber in the z direction is 2π, while the minimum wavenumber in the x direction is $2\pi/\hat{L}_x$. So, finally, the condition $\Lambda_1 < 1$ for energy stability implies a condition on the Reynolds number :

$$\mathrm{Re} < \mathrm{Re_E} = \frac{16\pi^2}{\max(L_x^2/L_z^2, 1)} \tag{6.66}$$

This shows that large enough viscosity (low enough Reynolds number) can always stabilize a shear flow.

The implication of this result for a square periodic domain, for instance, is that the maximum Reynolds number for stability[1] of a linear shear flow is $\mathrm{Re}_E = 16\pi^2 \simeq 158$. For larger Reynolds numbers, we know that the flow is linearly stable, but well-chosen finite amplitude instabilities can destabilize it. The question of the *optimal* perturbations, i.e. for a given perturbation energy, what is the shape of the perturbation that is unstable for the lowest possible Reynolds number, is a subject of active research.

[1]This is another good way of testing the numerics of a doubly-periodic code : for any $\mathrm{Re} < \mathrm{Re}_E$ the total energy of any initial perturbation should decay.