2.5 Sound waves in an inhomogeneous, time-dependent medium

So far, we have only dealt with cases where c was constant. This, however, is usually not true in most realistic systems. In what follows, we now apply the wave-packet approximation to study waves in a medium with non-constant c. Before we start, we first have to re-derive the governing equation in this more general case.

2.5.1 Derivation of the governing equation

Consider (for instance) a planetary or stellar atmosphere where the temperature profile is known, and given by $\bar{T}(z,t)$ (where z denotes the vertical coordinate – here, we assume for simplicity that there are no horizontal gradients of temperature). Then, the pressure and density of that background state are related both by the equation of state and by hydrostatic equilibrium:

$$\begin{aligned} \bar{p}(z,t) &= p(\bar{\rho}(z,t), \bar{T}(z,t)) \\ \frac{\partial \bar{p}}{\partial z} &= -\bar{\rho}(z,t) g \end{aligned} \tag{2.1}$$

Finally, if we want to assume that the background state does not undergo any fluid motions, then we also need to assume that $\partial\bar{\rho}/\partial t = 0$. One could, in principle, solve these equations although for our purposes we do not need to do so right now – we merely need to know solutions exist and govern the background state. As before, we next consider perturbations to this background state, such that $p(x,y,z,t) = \bar{p}(z,t) + \tilde{p}(x,y,z,t)$ (and similarly for T and ρ). Plugging this into the governing system of compressible equations, linearizing, and neglecting the effect of gravity on the perturbations themselves we have

$$\begin{aligned} \frac{\partial \tilde{\rho}}{\partial t} + \nabla \cdot (\bar{\rho} \tilde{\boldsymbol{u}}) &= 0 \\ \bar{\rho} \frac{\partial \tilde{\boldsymbol{u}}}{\partial t} &= -\nabla \tilde{p} \end{aligned} \tag{2.2}$$

so following the same steps as before we have

$$\frac{\partial^2 \tilde{\rho}}{\partial t^2} = -\frac{\partial}{\partial t} \nabla \cdot (\bar{\rho} \tilde{\boldsymbol{u}}) = \nabla^2 \tilde{p} \tag{2.3}$$

As discussed at the beginning of this chapter, in order to relate $\tilde{\rho}$ to $\tilde{p}$, one usually has to make further assumptions. Here we will assume that the perturbations are adiabatic, so

$$\frac{\partial \tilde{p}}{\partial t} = c_s^2 \frac{\partial \tilde{\rho}}{\partial t} \tag{2.4}$$

However, we now allow c_s (simply called c hereafter) to vary slowly with time and space. This then implies

$$\frac{\partial}{\partial t} \left(\frac{1}{c^2} \frac{\partial \tilde{p}}{\partial t} \right) = \nabla^2 \tilde{p} \tag{2.5}$$

We see that, if c only depends (slowly) on space, then we can take it out of the time-derivative and put it in the RHS to recover exactly the standard wave equation. However, if c depends (slowly) on time, then we cannot do this, and must keep c within the time derivative as is.

2.5.2 The wave packet equations

In general, the wave equation with non-constant c does not have simple analytical solutions. For example, d'Alembert's solution does not apply to the infinite problem, and sines and cosines are no longer eigenmodes of the finite problem in Cartesian coordinates.

However, if the function c varies *slowly* with position and time, as we have assumed here, we can go quite far by considering as before approximate solutions in the form of wave packets. This idea is essentially the same as using the WKB approximation, cf. AMS212B. Let's work through the same steps as in the previous section, but this time without assuming that c is constant. First, construct the wave packet, $\tilde{p} = Ae^{i\theta}$, and define the frequency and wavenumber as in the homogeneous case. Plugging this into the wave equation, we successively have

$$\frac{\partial \tilde{p}}{\partial t} = -i\omega A e^{i\theta} + \epsilon \frac{\partial A}{\partial T} e^{i\theta} \tag{2.6}$$

and

$$\frac{\partial}{\partial t}\left(\frac{1}{c^2}\frac{\partial \tilde{p}}{\partial t}\right) = -i\epsilon \frac{\partial \omega}{\partial T}\frac{A}{c^2} e^{i\theta} - 2i\epsilon \frac{\omega}{c^2}\frac{\partial A}{\partial T} e^{i\theta} - i\omega A \frac{\partial}{\partial T}\left(\frac{1}{c^2}\right) e^{i\theta} - \frac{\omega^2}{c^2} A e^{i\theta} \tag{2.7}$$

Meanwhile, $\nabla^2 \tilde{p}$ was already calculated in the previous section. To the lowest order, we then recover the dispersion relation as expected, $\omega^2 = c^2 k^2$ (where c now depends both on $\boldsymbol{X}$ and T) and to the next order we have

$$\frac{\partial A}{\partial T} + \frac{c^2}{\omega}\boldsymbol{k} \cdot \nabla_\epsilon A = -\frac{Ac^2}{2\omega}\left[\frac{\partial}{\partial T}\left(\frac{\omega}{c^2}\right) + \nabla_\epsilon \cdot \boldsymbol{k}\right] \tag{2.8}$$

As before, we then use the dispersion relation to find an alternative, more intuitive set of governing equations for $\boldsymbol{k}$ and ω "following the wave packet". Note that this time, the dispersion relation depends on time both explicitly, via $c(\boldsymbol{X}, T)$, and implicitly, via $\boldsymbol{k}$. Let's rewrite it as

$$\omega = \Omega(\boldsymbol{k}(\boldsymbol{X}, T), \boldsymbol{X}, T) \tag{2.9}$$

Hence

$$\begin{aligned} \frac{\partial \omega}{\partial T} &= \left(\frac{\partial \Omega}{\partial T}\right)_{\boldsymbol{k}} + \frac{\partial \Omega}{\partial k_x}\frac{\partial k_x}{\partial T} + \frac{\partial \Omega}{\partial k_y}\frac{\partial k_y}{\partial T} + \frac{\partial \Omega}{\partial k_z}\frac{\partial k_z}{\partial T} \\ &= \left(\frac{\partial \Omega}{\partial T}\right)_{\boldsymbol{k}} - \frac{\partial \Omega}{\partial k_x}\frac{\partial \omega}{\partial X} - \frac{\partial \Omega}{\partial k_y}\frac{\partial \omega}{\partial Y} - \frac{\partial \Omega}{\partial k_z}\frac{\partial \omega}{\partial Z} \end{aligned} \tag{2.10}$$

This implies that if we define a *group velocity* vector as

$$\boldsymbol{c}_g = (\partial\Omega/\partial k_x, \partial\Omega/\partial k_y, \partial\Omega/\partial k_z) \tag{2.11}$$

we can rewrite the evolution equation for ω much more concisely as

$$\frac{\partial\omega}{\partial T} + \boldsymbol{c}_g \cdot \nabla_\epsilon \omega = \left(\frac{\partial\Omega}{\partial T}\right)_{\boldsymbol{k}} \tag{2.12}$$

Note that, for the dispersion relation specific to the wave equation, we can write

$$\omega^2 = c^2 \boldsymbol{k}\cdot\boldsymbol{k} \rightarrow \boldsymbol{c}_g = \frac{c^2}{\omega}\boldsymbol{k} \tag{2.13}$$

so the same group velocity does, in fact, appear in the equation for the amplitude of the wave packet (2.8) – this is of course not a coincidence.

Similarly, we can work on the evolution of $\boldsymbol{k}$. For simplicity, let's just consider the $x-$component of the $\boldsymbol{k}$ vector (the other components are treated exactly the same way). Starting from $\frac{\partial \boldsymbol{k}}{\partial T} = -\nabla_\epsilon \omega$ we have

$$\frac{\partial k_x}{\partial T} = -\frac{\partial\omega}{\partial X} = -\left(\frac{\partial\Omega}{\partial X}\right)_{\boldsymbol{k}} - \frac{\partial\Omega}{\partial k_x}\frac{\partial k_x}{\partial X} - \frac{\partial\Omega}{\partial k_y}\frac{\partial k_y}{\partial X} - \frac{\partial\Omega}{\partial k_z}\frac{\partial k_z}{\partial X} \tag{2.14}$$

As before, we use the fact that $\nabla \times \boldsymbol{k} = 0$ (see previous lecture) to rewrite this as

$$\frac{\partial k_x}{\partial T} = -\left(\frac{\partial\Omega}{\partial X}\right)_{\boldsymbol{k}} - \frac{\partial\Omega}{\partial k_x}\frac{\partial k_x}{\partial X} - \frac{\partial\Omega}{\partial k_y}\frac{\partial k_x}{\partial Y} - \frac{\partial\Omega}{\partial k_z}\frac{\partial k_x}{\partial Z} \tag{2.15}$$

so that this can be written more concisely as

$$\frac{\partial k_x}{\partial T} + \boldsymbol{c}_g \cdot \nabla_\epsilon k_x = -\left(\frac{\partial\Omega}{\partial X}\right)_{\boldsymbol{k}} \tag{2.16}$$

and similarly for the other components. We therefore have

$$\frac{\partial \boldsymbol{k}}{\partial T} + \boldsymbol{c}_g \cdot \nabla_\epsilon \boldsymbol{k} = -\left(\nabla_\epsilon \Omega\right)_{\boldsymbol{k}} \tag{2.17}$$

These three equations show that

- Both A, $\boldsymbol{k}$ and ω are advected around by the *group velocity vector* which can be constructed from the gradient of the dispersion relation in wavenumber space as in equation (2.11). The characteristics of these equations are all the same, and form the *ray paths* of the waves.
- If that dispersion relation is independent of time, then ω is conserved along a ray path (see Equation (2.12)).
- If the dispersion relation is independent of space entirely, then $\boldsymbol{k}$ is conserved along a ray path (see Equation (2.17)).
- If the dispersion relation is independent of one particular space-variable, then the projection of $\boldsymbol{k}$ on that variable is invariant along the ray path.

Although we have proved these statements in the context of a wave equation, they are in fact more general, and can be demonstrated to hold for *all* linear waves!

2.6 Application: Sound waves in stars & planets

Let's now apply our findings to study the propagation of sound waves just below the surface of a star, or that of a giant gaseous planet [1]. Suppose the sound-speed profile is $c(Z)$, where $Z = 0$ at the surface, and Z increases downward into the stellar/planetary interior. Let's ignore curvature for the moment, and simply model the near-surface layer using Cartesian coordinates. Suppose for instance that c increases linearly with Z as $c(Z) = c_0 + c_1 Z$ (this is not completely ad-hoc, as it could for example be the first few terms of a Taylor expansion for small Z). Both c_0 and c_1 are positive, for the sound speed to increase with depth below the surface.

Suppose that some phenomenon (a near-surface convective eddy, or material falling onto the surface, such as a comet, or a planet, etc..) located at $(X = 0, Z = 0)$ causes the excitation of a wave packet at the surface. For simplicity, we will assume that this perturbation is only 2-dimensional, though the same procedure can be carried out without much more difficulty in 3D. The expression for the pressure perturbation associated with the wave packet is

$$\tilde{p}(\boldsymbol{X}, t) = A(\boldsymbol{X}, T) \exp(i\theta(\boldsymbol{x}, t)) \tag{2.18}$$

At time $t = 0$, we assume that the wave packet is localized (e.g. a $\delta-$function, or a narrow Gaussian), centered on $(0, 0)$. It has a uniform frequency ω_0 and a uniform wavevector $\boldsymbol{k}_0 = (k_{x0}, k_{z0})$. Note that in practice, there may be many waves excited, all with different $\boldsymbol{k}_0$; if that's the case then we would simply add them together. Figure 2.1 shows the model setup, as well as representative waves emitted.

Since the wave packet is propagating downward, towards *increasing* Z, the $z-$component of the group velocity must be positive. We can select the $x-$component of that velocity to be positive or negative arbitrarily, to select a left-going wave or a right-going wave. Let's take the positive case for instance. In that case, we are interested in the positive branch of the dispersion relation, and

$$\omega = c(Z)\sqrt{k_x^2 + k_z^2} \tag{2.19}$$

The group velocity is the gradient of this equation with respect to k_x and k_z, so

$$\boldsymbol{c}_g = c(Z)\left(\frac{k_x}{\sqrt{k_x^2 + k_z^2}}, \frac{k_z}{\sqrt{k_x^2 + k_z^2}}\right) = \frac{c(Z)}{|k|}\boldsymbol{k} \tag{2.20}$$

[1] In fact, this example can also apply to sound waves in a closed chamber on Earth, whose sound speed varies significantly with height, etc..

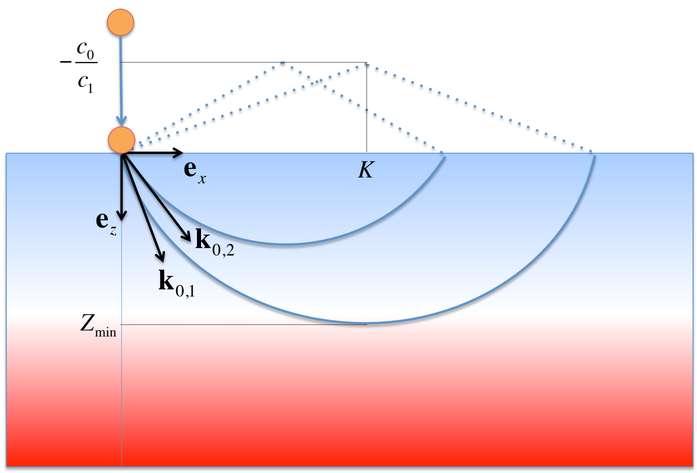

Figure 2.1: Examples of ray paths of waves emitted by a comet falling on the surface of a star or planet. Two representatives ray paths are shown. The initial wave vector $\boldsymbol{k}_{0,i}$ of the two waves has the same amplitude but $\boldsymbol{k}_{01}$ has a smaller k_{x0} and a larger k_{z0} while $\boldsymbol{k}_{02}$ has a larger k_{x0} and a smaller k_{z0}. In both cases, the ray paths are arcs of circles, and the centers of these circles are given in the text.

The equations for the evolution of the wave packet are

$$\begin{aligned}
&\omega(\boldsymbol{X},T)^2 = c(Z)^2|k(\boldsymbol{X},T)|^2 \\
&\frac{\partial \omega}{\partial T} + \boldsymbol{c}_g \cdot \nabla_\epsilon \omega = 0 \\
&\frac{\partial k_x}{\partial T} + \boldsymbol{c}_g \cdot \nabla_\epsilon k_x = 0 \\
&\frac{\partial k_z}{\partial T} + \boldsymbol{c}_g \cdot \nabla_\epsilon k_z = -\frac{dc}{dZ}|k| \\
&\frac{\partial A}{\partial T} + \boldsymbol{c}_g \cdot \nabla_\epsilon A = -\frac{A}{2kc(Z)}\left[\frac{\partial \omega}{\partial T} + c(Z)^2 \nabla_\epsilon \cdot \boldsymbol{k}\right]
\end{aligned} \tag{2.21}$$

We see that

- ω is invariant along a ray path

- k_x is invariant along a ray path, but k_z isn't.

We can first solve these equations for ω, k_x and k_z. The characteristic equations (which ultimately determine the ray paths) are the same for all quantities in the wave packet, and are given by:

$$\frac{\partial T}{\partial \tau} = 1$$
$$\frac{\partial X}{\partial \tau} = \boldsymbol{c}_g \cdot e_X = \frac{k_x}{\sqrt{k_x^2 + k_z^2}}$$
$$\frac{\partial Z}{\partial \tau} = \boldsymbol{c}_g \cdot e_Z = \frac{k_z}{\sqrt{k_x^2 + k_z^2}} \tag{2.22}$$

The quantities ω, k_x, k_z then satisfy, along a ray path

$$\frac{\partial \omega}{\partial \tau} = 0$$
$$\frac{\partial k_x}{\partial \tau} = 0$$
$$\frac{\partial k_z}{\partial \tau} = -\frac{dc}{dZ}|k| \tag{2.23}$$

The equation for A is not needed here, though could be solved as well if we want to.

Because ω and k_x are both conserved, $\omega = \omega_0$ and $k_x = k_{x0}$ along a ray path. So we can directly find k_z (along a ray path) from the dispersion relation without actually using the characteristic equations:

$$\omega_0 = c(Z)\sqrt{k_{x0}^2 + k_z^2} \rightarrow k_z = \sqrt{\frac{\omega_0^2}{c^2(Z)} - k_{x0}^2} \tag{2.24}$$

where we have selected the positive root for downward propagation. Note that one gets the *same* solution by solving simultaneously the equation $Z(\tau)$ and $k_z(\tau)$, and eliminating τ between the two... but that's a lot harder!

This immediately shows that the wave cannot propagate downward indefinitely, unless $k_{x0} = 0$; the lowest possible point of excursion is given by Z such that

$$c^2(Z_{\min}) = \frac{\omega_0^2}{k_{x0}^2} \rightarrow Z_{\min} = \frac{\frac{\omega_0}{k_{x0}} - c_0}{c_1} \tag{2.25}$$

This point is called the *lower turning point* of the wave. Note how $Z_{\min}$ decreases as k_{x0} increases – waves with larger horizontal wavenumber (smaller wavelengths) do not propagate as deeply as waves with smaller wavenumber (longer wavelengths). This is clearly seen in Figure 2.1.

To determine the shape of the ray path, we construct

$$\frac{dZ}{dX} = \frac{\frac{\partial Z}{\partial \tau}}{\frac{\partial X}{\partial \tau}} = \frac{k_z}{k_x} = \frac{\sqrt{\frac{\omega_0^2}{c^2(Z)} - k_{x0}^2}}{k_{x0}} = \sqrt{\frac{\omega_0^2}{k_{x0}^2 c^2(Z)} - 1} \tag{2.26}$$

which can be solved to get:

$$(c_0 + c_1 Z)^2 + c_1^2 (X + K)^2 = \frac{\omega_0^2}{k_{x0}^2} \tag{2.27}$$

where K is an arbitrary integration constant. For $Z = 0$ and $X = 0$ at $t = 0$ then we need K to satisfy

$$c_0^2 + c_1^2 K^2 = \frac{\omega_0^2}{k_{x0}^2} \tag{2.28}$$

so that

$$K = \pm\frac{1}{c_1}\sqrt{\frac{\omega_0^2}{k_{x0}^2} - c_0^2} \tag{2.29}$$

The ray path equation can be then rewritten as

$$\left(Z + \frac{c_0}{c_1}\right)^2 + (X + K)^2 = \frac{\omega_0^2}{c_1^2 k_{x0}^2} \tag{2.30}$$

We therefore see that the ray paths are circular, with radius ω_0/c_1k_{x0}, and centered vertically around the point $(-K, -c_0/c_1)$. The arbitrariness in the $\pm$ sign for K can finally be resolved by noting that we are seeking waves propagating to the right, so that K has to be negative. We see that the center of this circle is actually above the surface of the star, at $Z = -c_0/c_1$ (which is negative since both c_0 and c_1 are positive). The height of the center is independent of the initial conditions, but its X position is.

Once the packet reaches the lower turning point, it is *refracted*. The wave is evanescent beyond $Z_{\rm min}$, and must turn over. k_z changes sign, though it is easy to show that this does not change the equation for the ray path. It is worth noting that a refraction, however, does change the *phase* of the wave. To understand where this may come from, first note that the wave-packet solution is not really well-defined near the lower turning point. Indeed, one of the fundamental assumptions to the wave packet approximation is that the rate of variation of the background should be much slower than the wavelength of the oscillation. At the lower turning point, however, k_z tends to 0, so the vertical wavelength tends to infinity; the wave packet approximation breaks down.

To model the refraction correctly, one should therefore drop the wave-packet approximation in the limit $Z \to Z_{\rm min}$, find out what happens to the wave beyond $Z_{\rm min}$, and then match two solutions for Z above and below the turning point asymptotically to one another. The mathematical proof is fairly complex (see AMS212B for the tools needed to do it), but leads to the conclusion that the refracted wave is very much the same as the incoming wave, except for a phase shift. Interestingly, in the case of a refraction the phase shift is not $-\pi$, but $-\pi/2$ instead!

After the refraction, the wave packet returns to the surface; if the surface could be modeled as a boundary where $p = 0$, the wave would then be *reflected* there[2], then go back down, etc... Hence, the trajectory of the wave looks like

[2] This assumption is, in fact, not true in a real stellar photosphere since the latter is not a solid boundary. Instead, it turns out that the sound speed increases again very rapidly just above the photosphere, and there is another turning point – this time, an upper turning point. Hence the wave is also refracted near the photosphere, which leads to another $-\pi/2$ phase shift instead of a $-\pi$ phase shift. Since the derivative of c is very large, however, the radius of curvature at the upper turning point is very small, so the ray path looks very much like one for a reflected wave.

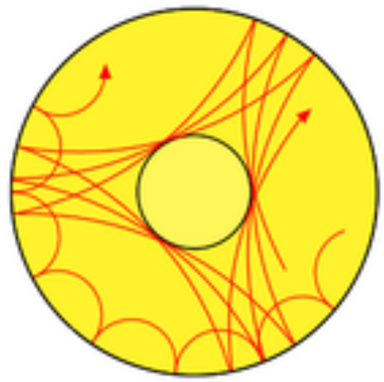

Figure 2.2: Examples of ray paths of waves in a star or planets, after several bounces/refractions. Image from Wikipedia.

the one shown in Figure 2.2 after several bounces near the surface and at the lower turning point.

This is not purely theoretical: for example, a rather spectacular collision between comet Shoemaker-Levy and Jupiter occurred in July 1994, and some of the images taken after impact at various wavelengths really show the position of the first bounce of the wave after impact.

This shows that a varying sound speed can create an acoustic cavity as well, whose depth depends on the horizontal wavenumber k_{0x} and frequency ω_0 of the wave. Because of this, stars have a *discrete* spectrum of oscillation frequencies, just like a musical instrument. This quantized spectrum can be reconstructed using similar arguments to the ones we used for waves in a 1D acoustic cavity. The frequencies are usually related to the sound travel time along a ray path – so by measuring these frequencies, it is often possible to reconstruct the sound speed profile below the surface of a star (cf. Helio/Asteroseismology).

2.7 Energy conservation

2.7.1 Energy conservation for the wave equation

In most systems that do no have explicit dissipation (viscosity, thermal dissipation, etc.), the energy is conserved. To see the principle of energy conservation at work, let's first see how the kinetic energy, $\bar{\rho}\tilde{u}^2/2$, evolves with time. Dotting the momentum equation with $\tilde{\boldsymbol{u}}$, we have

$$\bar{\rho}\frac{\partial}{\partial t}\left(\frac{\tilde{u}^2}{2}\right) = -\tilde{\boldsymbol{u}} \cdot \nabla\tilde{p} = -\nabla \cdot (\tilde{p}\tilde{\boldsymbol{u}}) + \tilde{p}\nabla \cdot \tilde{\boldsymbol{u}} \tag{2.31}$$

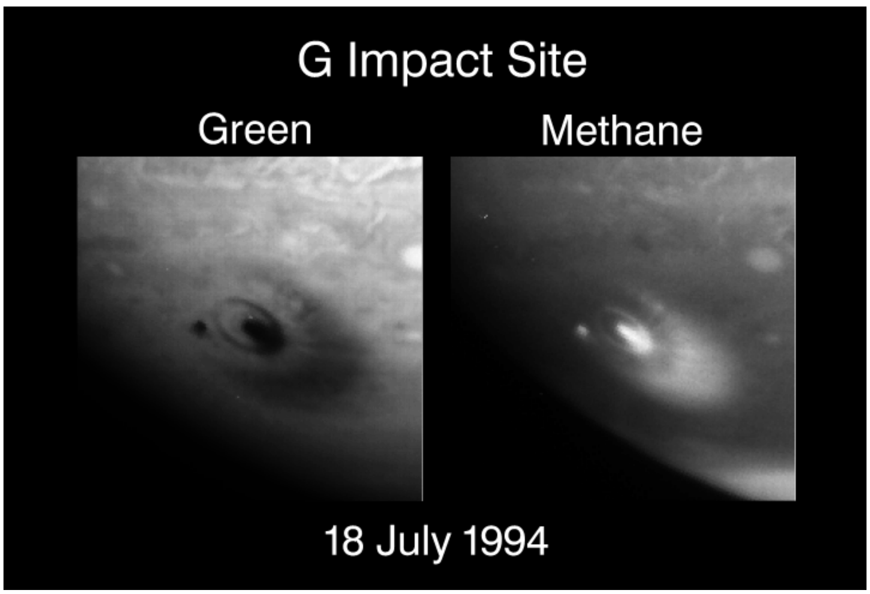

Figure 2.3: Images of Jupiter at different wavelengths, after impact with comet Shoemaker-Levy. The ring around the impact site shows the position of the first bounce of the main wave packet.

This last term can be rewritten using the mass conservation equation as

$$\nabla \cdot \tilde{\boldsymbol{u}} = \frac{1}{\bar{\rho}} \left[-\frac{\partial \tilde{\rho}}{\partial t} - \tilde{\boldsymbol{u}} \cdot \nabla \bar{\rho} \right] \tag{2.32}$$

so that

$$\bar{\rho} \frac{\partial}{\partial t} \left(\frac{\tilde{u}^2}{2} \right) = -\nabla \cdot (\tilde{p}\tilde{\boldsymbol{u}}) + \frac{\tilde{p}}{\bar{\rho}} \left[-\frac{1}{c^2} \frac{\partial \tilde{p}}{\partial t} - \tilde{\boldsymbol{u}} \cdot \nabla \bar{\rho} \right] \tag{2.33}$$

This can then be combined into

$$\bar{\rho} \frac{\partial}{\partial t} \left(\frac{\tilde{u}^2}{2} \right) + \frac{1}{\bar{\rho} c^2} \frac{\partial}{\partial t} \left(\frac{\tilde{p}^2}{2} \right) = -\frac{1}{\bar{\rho}} \nabla \cdot (\bar{\rho}\tilde{p}\tilde{\boldsymbol{u}}) \tag{2.34}$$

And finally

$$\bar{\rho} \frac{\partial}{\partial t} \left(\frac{\tilde{u}^2}{2} \right) + \frac{1}{\bar{\rho} c^2} \frac{\partial}{\partial t} \left(\frac{\tilde{p}^2}{2} \right) = -\frac{1}{\bar{\rho}} \nabla \cdot (\bar{\rho}\tilde{p}\tilde{\boldsymbol{u}}) \tag{2.35}$$

Recall that we had to assume that $\bar{\rho}$ is independent of time and only varies slowly with space, so

$$\frac{\partial}{\partial t} \left(\bar{\rho} \frac{\tilde{u}^2}{2} \right) + \frac{1}{c^2} \frac{\partial}{\partial t} \left(\frac{\tilde{p}^2}{2\bar{\rho}} \right) = -\nabla \cdot (\tilde{p}\tilde{\boldsymbol{u}}) \tag{2.36}$$

and finally,

$$\frac{\partial}{\partial t} \left(\bar{\rho} \frac{\tilde{u}^2}{2} + \frac{\tilde{p}^2}{2\bar{\rho} c^2} \right) = -\nabla \cdot (\tilde{p}\tilde{\boldsymbol{u}}) + \frac{\tilde{p}^2}{2\bar{\rho}} \frac{\partial}{\partial t} \left(\frac{1}{c^2} \right) \tag{2.37}$$

Now consider a particular fixed region of space D. Integrating this equation over D and recasting the first term on the RHS using the divergence theorem implies that

$$\frac{\partial E_D}{\partial t} = -\int_{\partial D} \tilde{p}\tilde{\boldsymbol{u}} \cdot \hat{\boldsymbol{n}} d^2x + \int_D \frac{\tilde{p}^2}{2\bar{\rho}} \frac{\partial}{\partial t} \left(\frac{1}{c^2} \right) d^3x \tag{2.38}$$

where

$$E_D = \int_D \left(\frac{\bar{\rho}}{2} \tilde{u}^2 + \frac{\tilde{p}^2}{2\bar{\rho} c^2} \right) d^3x \tag{2.39}$$

is the total energy of the wave inside the domain D, which has 2 terms : a kinetic energy, and a compressional energy. If the integral on the RHS is taken over a periodic domain, or in a closed container where $\tilde{\boldsymbol{u}} = \tilde{p} = 0$ on the boundary, then *the total energy is conserved unless there are temporal changes in the background state*. If c is independent of time, then

$$\frac{\partial E_D}{\partial t} = 0 \tag{2.40}$$

This tells us that, once a wave has been excited, if it is trapped in a closed container, or in a periodic domain, then its energy is conserved. In reality of course, dissipation would slowly take place, gradually eroding the wave's energy.

2.7.2 Energy conservation in the wave packet equation

Let's now go back to the wave packet amplitude equation. It is natural to hope that it also satisfies a similar energy conservation law as the original problem. As we can see, the energy is a function of $\tilde{\boldsymbol{u}}$ and $\tilde{p}$. We first need to express $\tilde{\boldsymbol{u}}$ in terms of the wave packet solution. Since

$$\bar{\rho}\frac{\partial \tilde{\boldsymbol{u}}}{\partial t} = -\nabla\tilde{p} \to -i\bar{\rho}\omega U e^{i\theta} = -i\boldsymbol{k}Ae^{i\theta} \tag{2.41}$$

to the lowest order in ϵ, and where U would be the equivalent amplitude of a wave packet in velocity. Multiplying each side of this equation by its complex-conjugate, we get $\bar{\rho}^2|U|^2 = |A|^2/c^2$ so that

$$E_K = \bar{\rho}|U|^2/2 = |A|^2/2\bar{\rho}c^2 = E_P \tag{2.42}$$

This means that the wave packet is a solution in which there is equipartition between the kinetic and acoustic energies, and the total energy being the sum of the two, we have

$$E = E_K + E_P = \frac{|A|^2}{\bar{\rho}c^2} \tag{2.43}$$

Note that here we always have to consider the norm of A squared, since A could be a complex quantity, but the energy has to be a real quantity.

Using the wave packet amplitude equation, we can now form the energy equation. We start by multiplying (2.8) by $A^*/\bar{\rho}c^2$ (where A^* is the complex conjugate of A), and rearranging:

$$\frac{1}{\bar{\rho}c^2}\frac{\partial|A|^2}{\partial T} + \frac{1}{\bar{\rho}c^2}\boldsymbol{c}_g\cdot\nabla_\epsilon|A|^2 = -\frac{|A|^2}{\bar{\rho}kc}\left[\frac{\partial}{\partial T}\left(\frac{\omega}{c^2}\right) + \nabla_\epsilon\cdot\boldsymbol{k}\right] \tag{2.44}$$

Then we construct the conservation law for $E = |A|^2/\bar{\rho}c^2$, with the intuition that energy, like all other quantities, must travel with the group velocity. To do this, we use the old trick of adding the same quantity on both sides, to put the LHS in the desired form:

$$\begin{aligned}\frac{\partial}{\partial T}\left(\frac{|A|^2}{\bar{\rho}c^2}\right) + \nabla_\epsilon\cdot\left(\boldsymbol{c}_g\frac{|A|^2}{\bar{\rho}c^2}\right) &= -\frac{|A|^2}{\bar{\rho}kc}\left[\frac{\partial}{\partial T}\left(\frac{\omega}{c^2}\right) + \nabla_\epsilon\cdot\boldsymbol{k}\right] \\ &+ |A|^2\frac{\partial}{\partial T}\left(\frac{1}{\bar{\rho}c^2}\right) + |A|^2\nabla_\epsilon\cdot\left(\frac{\boldsymbol{c}_g}{\bar{\rho}c^2}\right)\end{aligned} \tag{2.45}$$

Simplifying the RHS using the fact that $\bar{\rho}$ is constant (or at least, varies even slower than slow time or slow space), we get

$$\frac{\partial E}{\partial T} + \nabla_\epsilon\cdot(\boldsymbol{c}_g E) = -\frac{|A|^2}{\bar{\rho}}\left[\frac{1}{\omega c^2}\frac{\partial\omega}{\partial T} - \boldsymbol{k}\cdot\nabla_\epsilon\left(\frac{1}{kc}\right)\right] = -\frac{|A|^2}{\bar{\rho}\omega c^2}\left[\frac{\partial\omega}{\partial T} + \boldsymbol{c}_g\cdot\nabla_\epsilon\omega\right] \tag{2.46}$$

Using the evolution equation for ω, we then get

$$\frac{\partial E}{\partial T} + \nabla_\epsilon\cdot(\boldsymbol{c}_g E) = -\frac{|A|^2}{\bar{\rho}\omega c^2}k\frac{dc}{dT} = -\frac{|A|^2}{\bar{\rho}c^3}\frac{dc}{dT} = \frac{|A|^2}{2\bar{\rho}}\frac{d}{dT}\left(\frac{1}{c^2}\right) \tag{2.47}$$

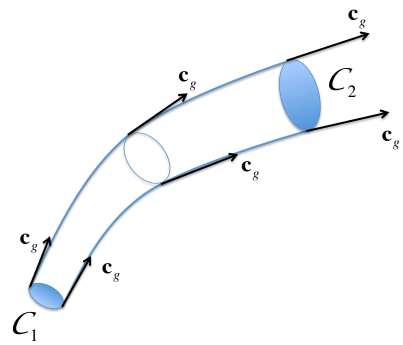

Figure 2.4: Example of a ray tube.

which is exactly what one would expect from the energy conservation equation discussed in the previous section (see Equation 2.38). This suggests that one could have written the amplitude equation for the wave packet *without doing any calculations at all* simply and directly from energy conservations considerations, as

$$\frac{\partial}{\partial T}\left(\frac{|A|^2}{\bar{\rho}c^2}\right) + \nabla_\epsilon \cdot \left(\boldsymbol{c}_g \frac{|A|^2}{\bar{\rho}c^2}\right) = \frac{|A|^2}{2\bar{\rho}} \frac{d}{dT}\left(\frac{1}{c^2}\right) \tag{2.48}$$

As discussed before, we see that the energy in the wave packet is conserved only if the sound-speed is time-invariant.

2.7.3 Conservation of wave energy along ray tubes

In many cases, the temporal variation of the background sound speed is negligible in comparison with the standard wave propagation timescale. In that case, the RHS of equation (2.47) is negligible, and we have

$$\frac{\partial E}{\partial T} + \nabla_\epsilon \cdot (\boldsymbol{c}_g E) = 0 \tag{2.49}$$

Consider again ray paths of the waves. These are also streamlines of the vector field $\boldsymbol{c}_g$, or in other words, lines that are tangent to $\boldsymbol{c}_g$ at any point in space. Now imagine a *ray tube*, whose surface is everywhere parallel to $\boldsymbol{c}_g$, as in the figure below. We now consider the volume $\mathcal{V}$ delimited on the sides by the ray tube, and on the ends by two cross sections $\mathcal{C}_1$ and $\mathcal{C}_2$, as shown in Figure **??**.

Integrating Equation (2.49) in $\mathcal{V}$, we get by the divergence theorem that

$$\frac{\partial}{\partial T}\int_{\mathcal{V}} E d^3X + \int_{\partial\mathcal{V}} E(\boldsymbol{c}_g \cdot \hat{\boldsymbol{n}}) d^2X = 0 \tag{2.50}$$

where $\hat{\boldsymbol{n}}$ is everywhere perpendicular to the surface of $\mathcal{V}$. Since $\boldsymbol{c}_g$ is everywhere parallel to the ray tube, then the contribution from the sides vanish ad we are left with the contributions from the flux through the areas $\mathcal{C}_1$ and $\mathcal{C}_2$:

$$\frac{\partial}{\partial T}\int_{\mathcal{V}} E d^3X = \int_{\mathcal{C}_2} E(\boldsymbol{c}_g \cdot \hat{\boldsymbol{n}}_2) d^2X + \int_{\mathcal{C}_1} E(\boldsymbol{c}_g \cdot \hat{\boldsymbol{n}}_1) d^2X \tag{2.51}$$

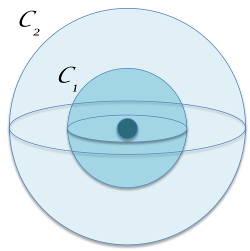

Figure 2.5: Spherical surfaces around the spherical loudspeaker. The energy flux through each spherical shell has to be the same.

where $\hat{\boldsymbol{n}}_1$ is the unit outward normal to $\mathcal{C}_1$, and similarly for $\hat{\boldsymbol{n}}_2$ (note that they are usually pointing in opposite directions). This shows that energy propagates along the ray tube without leaving through the sides.

This property is very useful if one wants to estimate the amplitude of a steady-state wave propagating in a ray tube of varying cross section. A steady-state implies that the LHS of the equation above is zero. This could happen for instance if the wave is trapped, or if there is a constant total energy flux into the tube on one side, and the same total energy flux out of the tube on the other. Then we simply have that

$$\int_{\mathcal{C}_2} E(\boldsymbol{c}_g \cdot \hat{\boldsymbol{n}}_2) d^2X = \int_{\mathcal{C}_1} E(\boldsymbol{c}_g \cdot (-\hat{\boldsymbol{n}}_1)) d^2X = 0 \tag{2.52}$$

for any two cross sections $\mathcal{C}_1$ and $\mathcal{C}_2$. This means that the total energy flux through *any* cross section of tube must be the same.

Worked example: the spherical loudspeaker

Let's go back to the example of the spherical loudspeaker. Instead of emitting a pulse of sound, let's now assume it is a constant source of it. Using energy conservation, show that $|A| \propto 1/R$.

Consider any two spherical surfaces centered at the speaker (see Figure 2.5) – by construction of the ray paths, these surfaces are perpendicular to $\boldsymbol{c}_g$, so $|\boldsymbol{c}_g \cdot \hat{\boldsymbol{n}}| = c_g = c$. By the results obtained above, we have that, for any sphere of radius R,

$$4\pi R^2 \frac{|A(R)|^2}{\bar{\rho} c} = \text{const} \tag{2.53}$$

so indeed, $|A| \propto 1/R$ as required. This is exactly the same result as we had found before, but this time in 3 lines of algebra!

2.8 Conclusions of this Chapter, and things to remember

This chapter introduced the wave packet as a universal tool for studying the propagation of non-dispersive waves in media where the wave propagation speed could depend slowly on time and space. Many of the results we derived also apply to dispersive waves, as we shall see in the next Chapter.

Given a wave packet defined as

$$p(\boldsymbol{x}, t) = A(\boldsymbol{X}, T)e^{i\theta(\boldsymbol{x},t)} \tag{2.54}$$

whose local frequency $\omega = -\partial\theta/\partial t$ and local wavenumber $\boldsymbol{k} = \nabla\theta$ are related through a dispersion relation $\omega = \Omega(\boldsymbol{k}; \boldsymbol{X}, T)$. Then

- The group speed of the packet is defined as

$$\boldsymbol{c}_g = (\partial\Omega/\partial k_x, \partial\Omega/\partial k_y, \partial\Omega/\partial k_z) \tag{2.55}$$

- The frequency within the wave packet evolves as

$$\frac{\partial \omega}{\partial T} + \boldsymbol{c}_g \cdot \nabla_\epsilon \omega = \left(\frac{\partial \Omega}{\partial T}\right)_{\boldsymbol{k}} \tag{2.56}$$

- The wave-vector within the wave packet evolves as

$$\frac{\partial \boldsymbol{k}}{\partial T} + \boldsymbol{c}_g \cdot \nabla_\epsilon \boldsymbol{k} = -\left(\nabla_\epsilon \Omega\right)_{\boldsymbol{k}} \tag{2.57}$$

- The evolution equation for the amplitude A of the wave-packet can be derived simply by considering energy conservation, and takes the form

$$\frac{\partial E}{\partial T} + \nabla_\epsilon \cdot (\boldsymbol{c}_g E) = \text{RHS} \tag{2.58}$$

  where the relationship between E and $|A|^2$, and the RHS of this equation, both depend on the system considered. In general, however, by Noether's theorem, we know that this RHS is zero if the medium considered is time-independent.