Homework 5

February 11, 2020

Do problem 1, and then *either* problem 2 or problem 3.

1 Local analysis for convection

Redo the local stability analysis for homogeneous convection we did in class, but this time add the dissipation terms we neglected: the viscous dissipation term in the momentum equation, and the thermal dissipation term in the temperature equation. What changes and what stays the same? Is λ still independent of $|k|$? What do the fastest-growing modes look like? You may assume that the system is 2D. It is not necessary, but will make your calculation easier.

2 Rayleigh Bénard convection with rotation (option 1)

This is a hard problem. See, e.g. Hydrodynamic Stability by Chandrasekhar if you need help.

Consider the Rayleigh-Bénard problem we studied in class (i.e. convection between bounded plates), with exactly the same setup. This question guides you towards finding a criterion for linear instability in the presence of rotation, when the rotation axis is in the same direction as gravity.

In the presence of rotation, oriented as in $\mathbf{\Omega} = (0, 0, \Omega)$, the governing equations become

$$\begin{aligned}
&\nabla \cdot \boldsymbol{u} = 0 \\
&\rho_m \left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + 2\mathbf{\Omega} \times \boldsymbol{u} \right) = -\nabla p + \rho \boldsymbol{g} + \rho_m \nu \nabla^2 \boldsymbol{u} \\
&\frac{\partial T}{\partial t} + \boldsymbol{u} \cdot \nabla T = \kappa_T \nabla^2 T \\
&\frac{\rho}{\rho_m} = -\alpha T
\end{aligned} \tag{1}$$

Question 1: Show that there exists a steady state with no fluid motions in which $\bar{T}(z) = T_m - z\Delta T/H$.

Question 2: Non-dimensionalize the equations exactly as in Section 5.2.6 of the notes. Show that an additional parameter $E = \frac{2\Omega H^2}{\kappa_T}$ appears in the presence of rotation.

Question 3: Assume 3D perturbations and write $T(x,y,z,t) = \bar{T}(z) + \tilde{T}(x,y,z,t)$. What system of equations do the perturbations satisfy? Linearize this system.

Question 4: Assume an ansatz of the form $q(x,y,z,t) = \hat{q}(z)e^{ilx+iky+\lambda t}$ for the modes, and substitute it into the equations. Show that the resulting equation for, say, $\hat{T}(z)$, is

$$(\lambda - \mathrm{Pr}D)^2(\lambda - D)D\hat{T} + (\lambda - \mathrm{Pr}D)(l^2+k^2)\mathrm{RaPr}\hat{T} + E^2(\lambda - D)\frac{d^2\hat{T}}{dz^2} = 0 \tag{2}$$

where D is an operator defined as $D \equiv \frac{d^2}{dz^2} - (k^2+l^2)$.

Question 5: Check that when $E = 0$ you recover the same equation as we had in the non-rotating case. Also check that modes with $\hat{T}(z) = \sin(n\pi z)$ are solutions of the equation. What equation does this imply for λ?

Question 6: Set $\lambda = 0$, and thus determine the critical value of needed for the onset of convection, as a function of n, k, l, and E. How does it differ from the non-rotating case? Does rotation inhibit or promote convection?

3 Double-diffusive convection (option 2)

This is also a hard problem, though you can find the answer or help with the answer in, e.g., Radko 2013, Double-diffusive convection, or some of my early papers on the topic on my website (e.g. Traxler et al. 2011)

Consider the following set of equations, which model homogeneous double-diffusive convection (ignoring the effects of boundaries). This is essentially the same as normal convection, except that there is a second scalar field C (for concentration of some solute) that contributes to the density.

$$\begin{aligned}
&\nabla \cdot \boldsymbol{u} = 0 \\
&\rho_m\left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\right) = -\nabla p + \rho\boldsymbol{g} + \rho_m\nu\nabla^2\boldsymbol{u} \\
&\frac{\partial T}{\partial t} + \boldsymbol{u}\cdot\nabla T = \kappa_T\nabla^2 T \\
&\frac{\partial C}{\partial t} + \boldsymbol{u}\cdot\nabla C = \kappa_C\nabla^2 C \\
&\frac{\rho}{\rho_m} = -\alpha T + \beta C
\end{aligned} \tag{3}$$

Question 1: Show that there exists a steady state with no fluid motions in which $\bar{T}(z) = T_m + zT_{0z}$, and $\bar{C}(z) = C_m + zC_{0z}$.

Question 2: In what follows, assume for simplicity that $T_{0z} > 0$ and $C_{0z} > 0$ (this selects the so-called fingering regime). Non-dimensionalize the equations using the unit length $[l] = d = \left(\frac{\kappa_T\nu}{\alpha g T_{0z}}\right)^{1/4}$, the unit velocity $[u] = \kappa_T/d$, the unit time $t = d^2/\kappa_T$, the unit temperature $[T] = dT_{0z}$ and the unit concentration $[C] = (\alpha/\beta)dT_{0z}$. (Ask me why this is a good choice in office hours if you are interested). Show that three interesting parameters pop up:

$$\mathrm{Pr} = \frac{\nu}{\kappa_T}, \quad \tau = \frac{\kappa_C}{\kappa_T}, \quad \text{and } R_0 = \frac{\alpha T_{0z}}{\beta C_{0z}} \tag{4}$$

Question 3: Assume 2D perturbations and write $T(x,z,t) = \bar{T}(z) + \tilde{T}(x,z,t)$ (and similarly for C). What system of equations do the perturbations satisfy? Linearize this system.

Question 4: Assume normal modes of the form $q(x,z,t) = \hat{q}e^{ilx+ikz+\lambda t}$, and substitute this ansatz into the equations. What equation does λ satisfy? It can be shown (though you do not have to do that!) that modes with $k = 0$ grow the fastest. What equation does λ satisfy for these modes?

Question 5: Set $\lambda = 0$ (in the equation for $k = 0$ modes), and thus determine the critical value of R_0 needed for the onset of fingering convection, as a function of l, and τ. Interpret your findings.