Homework 4

February 4, 2020

1 Problem 1: Gravity waves wave packet equations

Consider the governing equation for internal gravity waves (equation 4.7), in the case where N is a slowly varying function of $\boldsymbol{X}$ and the slow time τ. Assume a wave packet solution of the form

$$\phi = A(\boldsymbol{X}, \tau) e^{i\theta(\boldsymbol{x}, t)} \tag{1}$$

with $\boldsymbol{k} = \nabla\theta$ and $\omega = -\partial\theta/\partial t$.

Starting from these assumptions only, directly prove equations (4.23), (4.28), (4.29) of the lecture notes.

2 Problem 2: Gravity waves in a linearly stratified medium

Question 1: What are the ray paths of gravity wave packets in an ocean basin of uniform depth H that has $N = N_0$ (constant), where $Z = 0$ is the surface of the water, and Z increases downward? Note that you may assume that the basin is infinite in the horizontal direction. Plot an upward moving ray path starting from $X = 0$ and $Z = H$.

Question 2: Answer the same question, but with $N(Z) = aZ$.

3 Problem 3: Surface waves near the beach

Surface waves are vertical displacement of the surface of a fluid from its rest position. These displacements depend on the horizontal coordinates (x, y) and are usually denoted as $\eta(x, y, t)$. The dispersion relation for monochromatic surface waves oscillating at frequency ω, and with wavevector $\boldsymbol{k} = (k_x, k_y)$ is

$$\omega^2 = gk \tanh(hk) \tag{2}$$

(see AM107/217, for instance) where g is gravity, and h is the mean rest height of the water. The quantity h may be a slowly varying function of both x and y.

Question 1. Using only the dispersion relation for surface waves find out what the evolution equations for ω and $\boldsymbol{k} = (k_x, k_y)$ of a wave packet $\eta(x, y, t) = A(X, Y, T)e^{i\theta}$ would be.

Question 2. Consider a simple model in the (x, y) plane where there is a beach along the line $y = 0$, and the ocean is in the half-plane $y > 0$. The ocean floor away from the beach has a sloping bottom $h(x, y) = sy$ where s is the slope. A wave is coming towards the beach from far away, and is approaching it at an angle. Explain, using ray path theory, why the wave crests become parallel to the beach by the time the wave arrives at $y = 0$.