Homework 2 (Chapter 2 part 1)

January 15, 2020

Hand in 3 out of the 5 problems (Pick the ones you are most interested in).

1 Sound waves with gravity (part 1)

Question 1: Consider an isothermal atmosphere in a Cartesian coordinate system (see Chapter 1). Show that the equation for isothermal sound waves in that isothermal atmosphere, without neglecting gravity, is

$$\frac{\partial^2 \tilde{\rho}}{\partial t^2} = c^2 \nabla^2 \tilde{\rho} + g \frac{\partial \tilde{\rho}}{\partial z} \tag{1}$$

Question 2: Assume 1D monochromatic plane wave solutions to the wave equation derived above, assuming that $\tilde{\rho}$ only varies with z (i.e. ignore x and y dependences to only study $\tilde{\rho}(z,t)$)

- What is the dispersion relation?
- Based on the dispersion relation only, under which circumstances is the term that includes gravity negligible?
- Plug in typical dimensional numbers for g, c, k, etc.. Is gravity typically negligible for isothermal sound waves in air? Are there any circumstances in which it may not be negligible?

2 d'Alembert's solution

Solve the 1D Cartesian wave equation $\partial_{tt} p = c^2 \partial_{xx} p$ subject to initial conditions $p(x,0) = p_0 \exp(-x^2/2)$ and $\partial_t p(x,0) = 1$. How does this solution differ from the case where $\partial_t p(x,0) = 0$ studied in class? Sketch the wave packet evolution as a function of time to illustrate your argument (alternatively, plot it on the computer for $p_0 = 1$, $c = 1$, at carefully selected times, for instance).

3 Superposition of monochromatic waves vs. d'Alembert's solution

Solve the 1D Cartesian wave equation $\partial_{tt} p = c^2 \partial_{xx} p$ subject to initial conditions $p(x,0) = p_0 \exp(-x^2/2)$ and $\partial_t p(x,0) = 0$ using a superposition of monochromatic waves. How does this relate to d'Alembert's solution for the same initial conditions? Hint: I find that it helps to remember that, for any complex number, $\Re(z) = (z + z^*)/2$.

4 Global modes in a square

Find the 2D eigenmodes and eigenvalues of the wave equation $\frac{\partial^2 p}{\partial t^2} = c^2 \nabla^2 p$ in a square whose side length is 1, subject to $p = 0$ on the boundary of the region. Plot a few representative eigenmodes in each case. What are all the possible frequencies achievable?

5 Multiscale expansion for the damped oscillator

Problem 1:

- Consider the function $f(x) = e^{-\epsilon x}\sin(x)$. Calculate its derivative $\frac{df}{dx}$.
- Write $f(x)$ as a function of two variables $X_s = \epsilon x$ (the slow space variable), and $X_f = x$ (the fast space variable) as $f(X_s, X_f) = e^{-X_s}\sin(X_f)$. Compute

$$\frac{df}{dx} = \frac{\partial f}{\partial X_s}\frac{\partial X_s}{\partial x} + \frac{\partial f}{\partial X_f}\frac{\partial X_f}{\partial x} \tag{2}$$

  and verify that you recover the same answer answer as in the first case.

Problem 2: Consider the ODE

$$\frac{d^2 f}{dt^2} + f = -\epsilon \frac{df}{dt} \tag{3}$$

with initial conditions $f(0) = 1$ and $\frac{df}{dt}(0) = 0$. Write $f(t)$as the function $f(T_s, T_f)$ where $T_s = \epsilon t$ is the slow time and $T_f = t$ is the fast time. Compute $\frac{df}{dt}$, and $\frac{d^2 f}{dt^2}$, and use these to solve the ODE approximately, order-by-order in ϵ. Hint: you will find the answer (or close to the answer) in the AMS 212B lecture notes (see link in the Lecture page).