*Signal Processing*

*Letter*

# On the Reduction of Multidimensional DFT to Separabile DFT by Smith Normal form Theorem

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**Abstract.** The growing applications of image and image-sequence processing call for the computation of the discrete Fourier transform (DFT) of multidimensional signals defined on lattices of general type. The multidimensional DFT formula, introduced by Mersereau, allows one to choose the frequency domain sampling lattice, which is not univocally determinated by the signal definition lattice. On the other hand, the consistency of the inverse multidimensional DFT formula has been taken for granted in the general case. This work presents a proof based on the Smith normal form theorem.

## 1. INTRODUCTION

The discrete Fourier transform (DFT) is a useful rapresentation of finite size sequences, which is widely used for digital signal processing purposes, such as spectral analysis and fast convolution. As well known, given sequence $x(n)$ $n = 0, 1, ..., N-1$, its DFT is defined as

$$X(k)=\sum_{n=0}^{N-1} x(n)\,e^{-j2\pi kn/N},\quad k=0,1,\ldots,N-1$$

which corresponds to uniformly sampling one period of the Fourier transform of infinite sequence

$$\bar{x}(n)=\begin{cases} x(n), & n=0,1,\ldots,N-1 \\ 0 & \text{otherwise}\end{cases}$$

The properties of the DFT, and the expression for the inverse DFT formula, rely on the orthogonality of sequences $\{e^{-j2\pi kn/N}\}$. In fact, the following relationship holds

$$\sum_{k=0}^{N-1} e^{-j2\pi kn/N} = \begin{cases} N, & n=\ell N \\ 0, & \text{otherwise}\end{cases}$$

which gives raise to the inversion formula.

The one-dimensional DFThas been extended to the multidimensional case by Peterson [1], with more generality, and by Mersereau and Speake [2]. Such a concept embeds a degree of freedom not found in the one-dimensional case, namely the choice of the frequency domain sampling lattice supporting the DFT samples, which is not univocally determined by the signal definition lattice. The multidimensional (MD) DFT has revealed itself as a powerful tool for the analysis of TV signals (for example [3]). It is also a promising technique for the realization of efficient architectures for filtering signals defined on general lattices. Similarly to the $1D$ case, the properties of the MD DFT rely on the orthogonality of some multidimensional sequences (described in section 2). Such a property has been proved, for the two-dimensional case, by Ansari and Lee [4], which made use of the decomposition of a lattice into a finite number of cosets.

This work presents a new proof of the orthogonality of such MD sequences (and therefore of the consistency of multidimensional DFT), based on the Smith normal form theorem. The proof is simpler than Ansari's one, and the result holds for signals of any dimension. Besides, it suggests the method proposed by Mersereau [5] for computing the MD DFT, making it equivalent to a sequence of $1D$ DFT.

Section 2 introduces the notation and the background necessary to present the issues discussed in this note. In section 3, the key result is proved by means of the Smith normal form theorem. The consistency of the inverse MD DFT formula [2] is then verified as a simple corollary. Section 4 has the concluding remarks.

## 2. NOTATION AND PROBLEM STATEMENT

Let $u_c(x)$, $x \in \mathcal{R}^M$, be an $M$-dimensional signal with finite support $S_c$,i.e.,

$u_c(x) = 0$ for $x \notin S_c$

Assume $u_c(x)$ is sampled on lattice $\Lambda$ with basis $V$ [6] and call

$$u(x) = u_c(x) \tag{1}$$

for $x = Vn$, $n \in Z^M$.

The Fourier transform of discrete signal $u(x)$ is defined as

$$\begin{aligned} U(f) &= \sum_{x\in\Lambda} u(x)\exp\left(-j2\pi f^T x\right) \\ &= \sum_{n\in Z^M} u(Vn)\exp\left(-j2\pi f^T Vn\right),\ f\in\mathcal{R}^M \end{aligned} \tag{2}$$

and it is periodic with periodicity lattice equal to the reciprocal lattice $\Lambda^*$ with basis $(V^{-1})^T$.

The periodicization of $u(x)$ with periodicity lattice $\Gamma$ with basis $\boldsymbol{B} = \boldsymbol{VN}$ (note that $\Gamma$ must be a sublattice of $\Lambda$, hence $N$ must be an integer full rank matrix) gives

$$\bar{u}(x) = \sum_{n\in Z^M} u(x+Bn) \tag{3}$$

Clearly, signal $u(x)$ can be recovered from $\bar{u}(x)$ if and only if there exists an elementary cell [6] $\bar{I}_B$ of $\Gamma$ such that

$$\bar{I}_B \supseteq S_c \tag{4}$$

Condition (4) is thereafter referred to as the "correct periodicization" condition, and throughout the work it is assumed to be always satisfied.

It is convenient to introduce the following definition: given $M$-dimensional lattices $\Lambda$ e $\Gamma$, with $\Gamma \subseteq \Lambda$, and any elementary cell $I$ of $\Gamma$, the set $\Lambda \cap I$ will be called a $\Gamma$-period of $\Lambda$. Clearly, the number of points in any $\Gamma$-period of $\Lambda$ is equal to the index of $\Gamma$ in $\Lambda$ [6]. Let us recall that the index of $\Gamma$ in $\Lambda$ is the number of cosets in quotient group $\Lambda/\Gamma$.

The Fourier transform $\bar{U}(f)$ of $\bar{u}(x)$ is the sampled version of $U(f)$ on the reciprocal lattice $\Gamma^*$ with basis $(B^{-1})^T$:

$$\bar{U}(f) = U(f) \quad \text{for } f = \left(B^{-1}\right)^T k,\quad k\in Z^M \tag{5}$$

It is immediate to see that

$$\begin{aligned} U\left[\left(B^{-1}\right)^T k\right] &= \sum_{x\in I_B} \bar{u}(x)\exp\left(-j2\pi k^T B^{-1} x\right) \\ &= \sum_{Vn\in I_B} \bar{u}(Vn)\exp\left(-j2\pi k^T B^{-1} Vn\right) \\ &= \sum_{Vn\in I_B} \bar{u}(Vn)\exp\left(-j2\pi k^T N^{-1} n\right) \end{aligned} \tag{6}$$

where $I_B$ is any $\Gamma$-period of $\Lambda$

Let

$$\bar{u}_V(n) = \bar{u}(Vn), \qquad n \in Z^M \tag{7}$$

and

$$U_{\mathrm{DFT}}(k) = \bar{U}\left[\left(B^{-1}\right)^T k\right], \quad k\in Z^M \tag{8}$$

Signal $\bar{u}_V(n)$ is periodic with periodicity lattice basis $N$, and signal $U_{\mathrm{DFT}}(k)$ is periodic with periodicity lattice basis $N^T$.

Let $\psi$ be the lattice with basis $N$ and $I_N$ any $\psi$-period of $Z^M$. Equation (6) from (7) and (8) can be written as

$$U_{\mathrm{DFT}}(k) = \sum_{n\in I_N} \bar{u}_V(n)\exp\left(-j2\pi k^T N^{-1} n\right) \tag{9}$$

which represents the discrete Fourier transform of a sampled periodical signal [2]. In expression (9) both signal and DFT are supported by orthogonal lattices, a remarkable feature. They play the role of "normalized" lattices.

Note that the number of terms in summation (9) is equal to the number of points of $I_N$, which is $|\det(N)|$.

As an example, Fig 1a) shows a 2D signal sampled on lattice with basis

$$V = \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix}$$

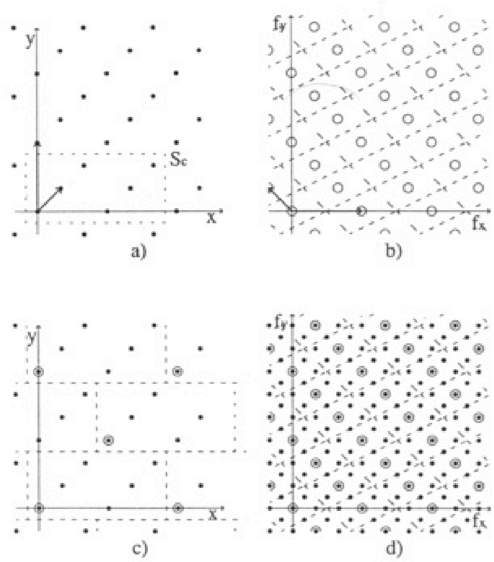


**Fig. 1** - Spatial sampling and periodicization and corresponding ef-

The sampling lattice is shown with filled dots and the region inside the dashed line of Fig. 1a) gives the support $S_c$. In Fig 1b) is shown the periodicization of Fourier transform of signal of Fig. 1a) on lattice

$$V^{-T} = \begin{bmatrix} 1/3 & 0 \\ -1/3 & 1 \end{bmatrix}$$

Repetition centers are shown with empty dots and dashed lines indicate a possible division of plane with elementary cells. In Fig 1c) and 1d) one can see periodicization of Fig 1a) on lattice with basis

$$P = \begin{bmatrix} 3 & 6 \\ 3 & 0 \end{bmatrix}$$

and the corresponding sampling of Fourier transform on lattice of basis

$$P^{-T} = \begin{bmatrix} 0 & 1/6 \\ 1/3 & -1/6 \end{bmatrix}$$

## 3. THE MULTIDIMENSIONAL INVERSE DFT FORMULA

Now we state the following proposition, which is instrumental for obtaining the $M$-dimensional inverse DFT expression [2]

*Proposition* 1

Let $N$ be any $M \times M$ integral full rank matrix and call $\psi$ the lattice with basis $N$ and $\Xi$ the lattice with basis $N^T$. Then the sequence $\{\exp(j2\pi k^T N^{-1} n)\}$, $n \in Z^M$, $k \in J_N$, where $J_N$ is any $\Xi$-period of $Z^M$, is orthogonal in the following sense

$$\sum_{k \in J_N} \exp\left(j 2\pi k^T N^{-1} n\right) = \begin{cases} |\det(N)|, & n \in \psi \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

Before proving the Proposition it is useful to report Smith normal form theorem [7], which is an essential tool for the proof.

Given an $M \times M$ matrix $T$, full rank and with integer entries, it is always possible to find two integer unimodular $M \times M$ matrices $S$ and $U$ (i.e., with $|\det(S)| = |\det(U)| = 1$) such that $STU = D$ with $D$ in Smith normal form, [7] i.e.,

i) $D = \text{diag}\{d_1, d_2, \ldots, d_M\}$
ii) $d_i > 0$, $1 \le i \le M$
iii) $d_i$ divides $d_j$, $1 \le i < j \le M$.

The last property, which guarantees the uniqueness of the Smith normal form, is not used in this work.

*Proof of proposition* 1

Observing that sequence $\{\exp(j2\pi k^T N^{-1} n)\}$ is periodic in $k$ on the points of lattice $\Xi$, i.e.,

$$\exp\left[j 2\pi (k+a)^T N^{-1} n\right] = \exp\left(j 2\pi k^T N^{-1} n\right) \quad \text{if} \quad a \in \Xi \tag{11}$$

one needs to prove the proposition for just one choice of $J_N$, the $\Xi$-period of $Z^M$.

Suppose $N = \text{diag}(N_1, N_2, \ldots, N_M)$, $N_i > 0$. Then $\psi = \Xi = Z(N_1) \times Z(N_2) \times \ldots \times Z(N_M)$, where $Z(N_i) \triangleq \{N_i \ell, \ell \in Z\}$ and "$\times$" means the cartesian product. Choosing set $([0, N_1) \times [0, N_2) \times \ldots \times [0, N_M)) \cap Z^M$ as $J_N$, it is easy to prove the proposition. In fact, putting $n = (n_1, n_2, \ldots, n_M)^T$ and $k = (k_1, k_2, \ldots, k_M)^T$, expression (10) becomes

$$\sum_{0\, n_1}^{N_1 - 1} \sum_{0\, n_2}^{N_2 - 1} \cdots \sum_{0\, n_M}^{N_M - 1} \exp\left(j 2\pi n_1 k_1 / N_1\right) \exp\left(j 2\pi n_2 k_2 / N_2\right) \cdots \exp\left(j 2\pi n_M k_M / N_M\right) = \begin{cases} N_1 N_2 \ldots N_M = \det(N), & n_1 = N_1 \ell_1,\ n_2 = N_2 \ell_2, \ldots, n_M = N_M \ell_M \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

The general case can be managed by means of the Smith normal form theorem. Suppose $H$ and $L$ are unimodular matrices such that $\bar{N} = HNL$ is in Smith normal form, and call $\Phi$ the lattice with basis $\bar{N} = \bar{N}^T$. The l.h.s. of expression (10) becomes

$$\sum_{k \in J_N} \exp\left(j 2\pi k^T L \bar{N}^{-1} H n\right) \tag{13}$$

It is easy to prove that the set $J_{\bar{N}} = \{L^T k | k \in J_N\}$ is a $\Phi$-period of $Z^M$, and that $\{Hn | n \in \psi\} = \Phi$. Putting $L^T k = \bar{k}$, $Hn = \bar{n}$, expression (10) is equivalent to

$$\sum_{\bar{k} \in J_{\bar{N}}} \exp\left(j 2\pi \bar{k}^T \bar{N}^{-1} \bar{n}\right) = \begin{cases} \left|\det\left(\bar{N}\right)\right|, & \bar{n} \in \Phi \\ 0 & \text{otherwise} \end{cases} \tag{14}$$

Identity (14) is verified, since $\bar{N}$ is diagonal. Therefore the proposition is proved •.

It is then possible to state the inverse DFT formula [2]. With the notation used in Section 2, the following equality holds

$$\bar{u}_v(n) = \frac{1}{|\det(N)|} \sum_{k \in J_N} U_{\mathrm{DFT}}(k) \exp\left(j\, 2\pi\, k^T N^{-1} n\right) \quad (15)$$

as it can be straightforwardly verified by means of Proposition 1.

## 4. CONCLUSION

The orthogonality of the multidimensional complex exponential sequences entering the multidimensional DFT on general lattices has been proved by using the Smith normal form theorem. By means of such a result, the consistency of the inverse MD DFT formula has been proved in the case of general frequency domain sampling lattices.

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