Express Letters

On the Determination of All the Sublattices of Preassigned Index and Its Application to Multidimensional Subsampling

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Abstract—**Subsampling offers an effective and simple way to implement a data compression technique. Its use in the video context is rather attractive, as its application to current video broadcasting schemes, such as MUSE and HD-MAC, clearly shows. A sensible exploitation of subsampling's potential requires the systematic evaluation of the image degradation effects due to the specific subsampling lattice. Such a possibility is equivalent to the determination of all the sublattices of a given index within a lattice, a problem that can be solved, as the work shows, on the basis of the Hermite normal-form theorem. The practical use of the result in the subsampling context is exemplified for a data compression case.**

I. Introduction

Subsampling (or decimation) is a fundamental data compression operation whose importance is related to the fact that with band-limited signals it may not lead to any information loss, and the original signal may be recovered from the subsampled signal by a suitable interpolation procedure [1]. Subsampling is part of a number of compression schemes for video broadcasting (e.g., MUSE and HD-MAC) and it remains very attractive for the simplicity of its implementation. In the context of 2-D and 3-D sampled signals, the geometry of the supports allows for a great variety of sampling and subsampling lattices [2]–[6]. An extensive study of the multidimensional lattices that are more suitable to compress the television signal at the data rates of broadcasting interest is given in [5]. Such a study makes clear that in the case of multidimensional signals, both information loss and perceptual quality of the subsampling/interpolation procedure can be controlled not only by the pre- and post-filter characteristics (as in the case of the 1-D signals) but also by the choice of the subsampling structure [6].

The possibility of properly choosing the subsampling structure requires (and it is equivalent to) the knowledge of all the sublattices of preassigned index within a lattice, as the next section explains. This work shows that the determination of all the sublattices of preassigned index, a problem whose solution is not found in the signal processing literature, can be solved by means of the Hermite normal-form theorem [7]. The implications of these notions in the subsampling context are demonstrated by means of an example.

II. Notation

Given matrix $\mathbf{V} = [\mathbf{v}_1|\mathbf{v}_2|\cdots|\mathbf{v}_\mathbf{M}]$ formed by $\mathbf{M}$ linear independent vectors of $\mathbb{R}^M$, an $\mathbf{M}$-dimensional lattice is defined as the set

$$\Lambda = \left\{\mathbf{x}: \mathbf{x} = \sum_{i=1}^{\mathbf{M}} n_i\mathbf{v}_i,\ n_i \in \mathbb{Z}\right\}. \qquad (1)$$

Matrix $\mathbf{V}$ is called the basis of lattice Λ [3]. It is important to note that for a given lattice Λ there is an infinite number of bases generating Λ, as explained later in this section. The lattices have bases $\mathbf{V} \in \mathbb{Z}_\mathbf{M}$ (where $\mathbb{Z}_\mathbf{M}$ is the set of the $\mathbf{M} \times \mathbf{M}$ matrices with integer entries). Matrix $\mathbf{A} \in \mathbb{Z}_\mathbf{M}$ is unimodular if there exists a matrix $\mathbf{B} \in \mathbb{Z}_\mathbf{M}$ such that $\mathbf{AB} = \mathbf{BA} = \mathbb{I}$, where $\mathbb{I}$ is the identity matrix. One can easily prove that a matrix $\mathbf{A} \in \mathbb{Z}_\mathbf{M}$ is unimodular if and only if $|\det \mathbf{A}| = 1$. Matrixes $\mathbf{U} \in \mathbb{Z}_\mathbf{M}$ and $\mathbf{V} \in \mathbb{Z}_\mathbf{M}$ are right-equivalent if there exists a unimodular matrix $\mathbf{A}$ such that $\mathbf{V} = \mathbf{UA}$.

Matrixes $\mathbf{U}$ and $\mathbf{V}$ are bases of the same lattice if and only if they are right-equivalent [3]. Hence all the bases $\mathbf{V}$ of a given lattice share quantity $|\det \mathbf{V}|$; such a quantity is called the module of Λ and is denoted as $d(\Lambda)$.

Given two lattices Γ and Λ, lattice Γ is a sublattice of Λ if $\Gamma \subseteq \Lambda$. Quantity $d(\Gamma)/d(\Lambda)$ is called the index of Γ in Λ.

If Γ is a sublattice of Λ and $\mathbf{U}$ is a basis of Λ, then a basis $\mathbf{V}$ of Γ can be obtained as $\mathbf{V} = \mathbf{UA}$, with $\mathbf{A}$ a suitable matrix of $\mathbb{Z}_\mathbf{M}$ such that $|\det \mathbf{A}| = d(\Gamma)/d(\Lambda)$. Conversely, if $\mathbf{A} \in \mathbb{Z}_\mathbf{M}$, matrix $\mathbf{V} = \mathbf{UA}$ is the basis of a sublattice of Λ of index $|\det \mathbf{A}|$ in Λ.

Matrix $\mathbf{H} \in \mathbb{Z}_\mathbf{M}$, $\mathbf{H} = \{h_{ij};\ i = 1, 2, \cdots, \mathbf{M};\ j = 1, 2, \cdots \mathbf{M}\}$, is in Hermite normal form if a) it is upper triangular; b) $h_{ij} \geq 0$, $1 \leq i, j \leq \mathrm{M}$; and c) $h_{ij} < h_{ii}$, $1 \leq i < j \leq M$ [7]. (The above definition is not the most general; nevertheless, it suits the needs of this work). The set of all the matrixes of $\mathbb{Z}_\mathbf{M}$ in Hermite normal form is denoted as $\mathscr{H}_\mathbf{M}$.

III. Problem Statement and Solution

The above definitions allow us to formalize the problem presented in the introduction in the following terms: Given an $\mathbf{M}$-dimensional lattice Λ and a positive integer k, determine all the distinct sublattices of Λ with index k in Λ. Such a task, if $\mathbf{U}$ is a basis of Λ, corresponds to the partitioning of set

$$\mathbb{V} = \{\mathbf{V} \in \mathbb{Z}_\mathbf{M};\ \mathbf{V} = \mathbf{UA};\ \mathbf{A} \in \mathbb{Z}_\mathbf{M};\ |\det \mathbf{A}| = k\} \qquad (2)$$

into distinct classes $\mathbb{V}_i$ such that, if $\mathbf{B}$ and $\mathbf{C}$ are two matrixes of $\mathbb{V}$, they belong to the same class $\mathbb{V}_i$, i.e., it is $\mathbf{B} \in \mathbb{V}_i$ and $\mathbf{C} \in \mathbb{V}_i$ if and only if $\mathbf{B}$ and $\mathbf{C}$ are right-equivalent.

Such a problem is also equivalent to the partitioning of set

$$\mathbb{A} = \{\mathbf{A} \in \mathbb{Z}_\mathbf{M};\ |\det \mathbf{A}| = k\}$$

into distinct classes $\mathbb{A}_i$ such that if $\mathbf{E}$ and $\mathbf{F}$ are two matrixes of $\mathbb{A}$, $\mathbf{E} \in \mathbb{A}_i$ and $\mathbf{F} \in \mathbb{A}_i$ if and only if $\mathbf{E}$ and $\mathbf{F}$ are right-equivalent.

It is clear that each sublattice Γ_i of Λ with index k in Λ and basis $\mathbf{V}_i$ corresponds to just one class $\mathbb{A}_i$, and the relationship $\mathbf{V}_i = \mathbf{VA}$, $\mathbf{A} \in \mathbb{A}_i$ holds. Conversely, each class $\mathbb{A}_i$ corresponds to only one sublattice Γ_i of Λ with index k in Λ.

The last problem reformulation is especially useful in light of the fact that the Hermite normal-form theorem [7] states that every matrix $\mathbf{A} \in \mathbb{Z}_\mathbf{M}$ is right-equivalent to one and only one

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TABLE I
(A) MATRIXES OF $\mathscr{H}_2$ WITH DETERMINANT EQUAL TO 4;
(B) INVERSE TRANSPOSED OF THE MATRIXES OF (A)

(a)	$\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{4} & 0 \\ 0 & 1 \end{pmatrix}$
(b)	$\begin{pmatrix} 4 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{4} & 0 \\ -\frac{1}{4} & 1 \end{pmatrix}$
(c)	$\begin{pmatrix} 4 & 2 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{4} & 0 \\ -\frac{1}{2} & 1 \end{pmatrix}$
(d)	$\begin{pmatrix} 4 & 3 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{4} & 0 \\ -\frac{3}{4} & 1 \end{pmatrix}$
(e)	$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$
(f)	$\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2} & 0 \\ -\frac{1}{4} & \frac{1}{2} \end{pmatrix}$
(g)	$\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$

matrix of $\mathscr{H}_{\mathbf{M}}$. Hence the set of the Hermite normal-form matrixes with determinant equal to k.

$$\{\mathbf{H}_i \in \mathscr{H}_{\mathbf{M}}, |\det \mathbf{H}_i| = k\} \tag{3}$$

enjoys the property that each $\mathbf{H}_i$ belongs to just a single class $\mathbb{A}_i$; conversely, each class $\mathbb{A}_i$ of contains just a matrix $\mathbf{H}_i$ of (3).

It is worth observing that it is not straightforward to find a Hermite normal-form matrix $\mathbf{H}_i$ right-equivalent to a given matrix $\mathbf{A} \in \mathbb{Z}_{\mathbf{M}}$. Instead, it is trivial to find all the matrixes of $\mathscr{H}_{\mathbf{M}}$ with determinant equal to a fixed integer k simply using the definition given in Section I. As an example, all the matrixes of $\mathscr{H}_{\mathbf{M}}$ with determinant equal to four are reported in column (A) of Table I.

IV. APPLICATIONS TO MULTIDIMENSIONAL SUBSAMPLING

It is interesting to see the impact of the determination of all the sublattices of preassigned index in the context of multidimensional signal subsampling. The sublattice choice directly affects the possible aliasing resulting from the subsampling and the perceptual quality of a subsampling/interpolation chain, also in the case of theoretically adequate pre- and postfiltering [8].

Sampling a signal on lattice Λ of basis $\mathbf{V}$ implies the frequency domain repetition of its spectrum at the points of the dual lattice Λ^* of basis $\mathbf{V}^{-T}$ [3]. Hence, a preassigned data compression rate achieved by subsampling may lead to rather different results, depending on the adopted subsampling lattice. As an example, consider the 4:1 compression of the still image data of Fig. 1. As noted above, column (A) of Table I reports the bases of the seven sublattices obtainable by keeping a point out of four in a 2-D orthogonal lattice. The bases are the seven distinct 2×2 Hermite normal-form matrixes with determinant equal to four (note that their structure makes their determination straightforward). Column (B) of Table I shows the bases of the dual lattices of column (A).

The magnitude squared of the Fourier transform of Fig. 1 is shown in Fig. 2. The sublattices of index four in the orthogonal lattice with the matrixes of column (A) of Table I as bases are shown in column (A) of Fig. 3. The dark squares correspond to the position of the samples of the orthogonal lattice held by the sublattice, and the little circles to the position of the deleted samples. Column (B) of Fig. 3 shows the spectra of the subsampled versions of Fig. 1. It is very clear from Fig. 3 how the spectral superposition (i.e., the aliasing) due to subsampling strongly depends on the sublattice choice. For the spectrum of Fig. 1, sublattice (c) gives minimal superposition with respect to the other sublattices [see for instance sublattice (g)].

In the case of 3-D lattices, the number of sublattices of preassigned index can easily become large; for instance, in the 3-D case a data compression rate of four can be achieved by 35 different sublattices [7].

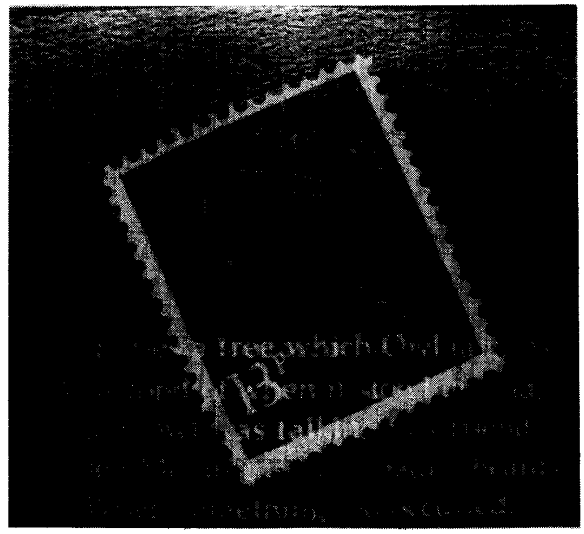

Fig. 1. Test image.

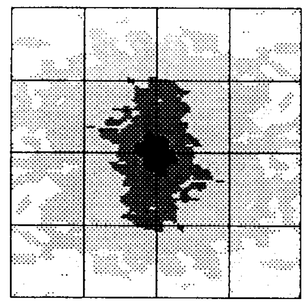

Fig. 2. Spectrum of the test image of Fig. 1.

V. CONCLUSION

The determination of all the sublattices of preassigned index in a lattice is necessary for a systematic approach of multidimensional subsampling. This work has shown that such a problem can be solved on the basis of the Hermite normal-form theorem. The complete list of the 2-D subsampling lattices of index four in the orthogonal lattice seen in Section III supplies an instructive example of the use of this result in the subsampling context. The example, in spite of its simplicity, clearly indicates the major role played by the sublattice choice in the image degradation due to the subsampling process.

In the case of video signals, the subsampling choices can easily become far more numerous and better articulated. Their system-

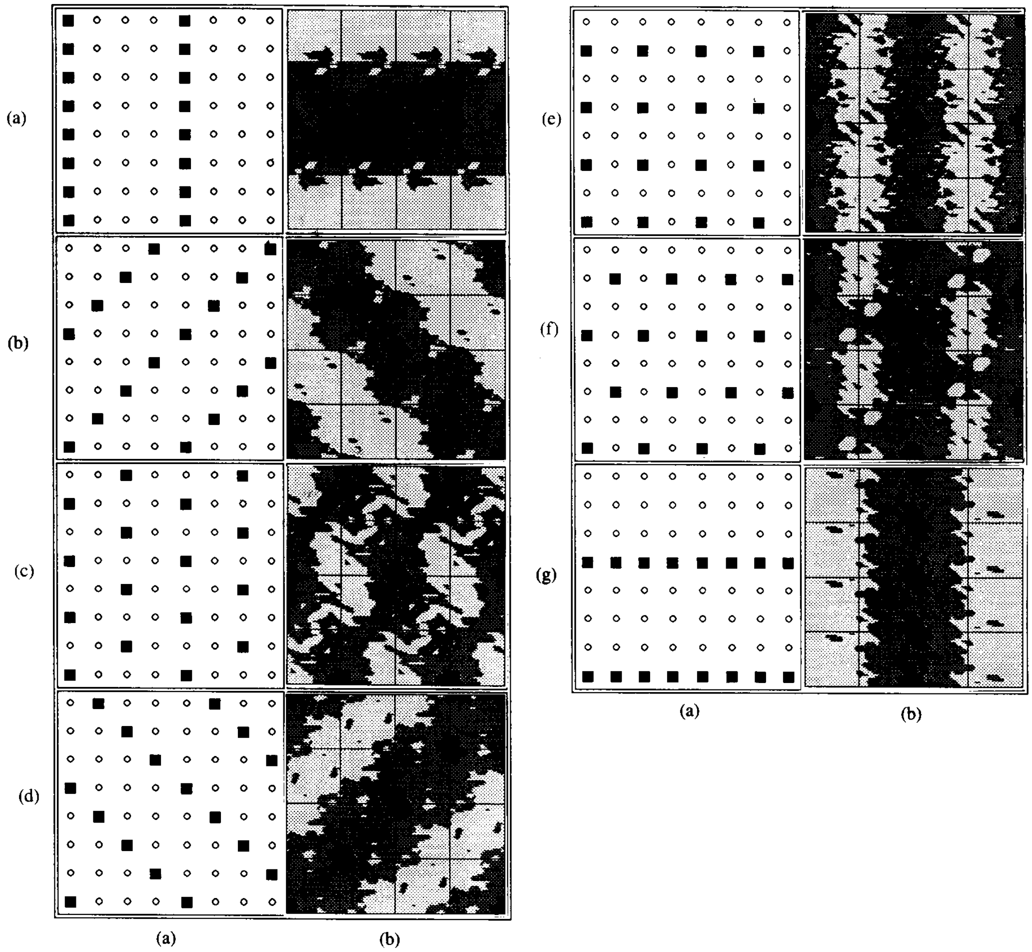

Fig. 3. (a) Sampling sublattices with the matrixes of column (A) of Table I as bases (■ = sublattice sample; ◦ = deleted sample); (b) Spectra of the test image subsampled on the sublattices of (A).

atic consideration, on the basis of the determination of all the sublattices of preassigned index presented in this work, enhances the possibility of exploiting the subsampling procedure's potential.

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HDTV Coding Using Hybrid MRVQ / DCT

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I. Introduction

Over the past few years the broadband integrated services digital network (B-ISDN), based on single-mode optical fiber technology, has been the focus of intense research. The B-ISDN is designed to handle a wide variety of services, ranging from voice to data and, ultimately, the distribution of high-quality

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